A NOTE ON PSEUDOCONVEXITY
AND PROPER HOLOMORPHIC MAPPINGS

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ABSTRACT. In this paper we discuss some connections between proper holomorphic mappings between domains in \( \mathbb{C}^n \) and the boundary behaviors of certain canonical invariant metrics. A compactness theorem has been proved. This generalizes slightly an earlier result proved by the second author.

Introduction. A continuous mapping \( f: X_1 \to X_2 \) between two topological spaces is called proper if \( f^{-1}(K) \subset X_1 \) is compact whenever \( K \subset X_2 \) is compact. Proper holomorphic mappings between analytic spaces stand out for their beauty and simplicity. For instance, if \( g: D_1 \to D_2 \) is a proper holomorphic mapping between two bounded domains in \( \mathbb{C}^n \), a theorem of Remmert says that \((D_1, g, D_2)\) is a finite branching cover. The branching locus in \( D_1 \) is described by \( \{z \in D_1 \mid \det(dg(z)) = 0\} \). For the past ten years, there has been a great amount of activity in characterizing the proper holomorphic mappings between pseudoconvex domains. It has been known for a long time that there are numerous proper holomorphic maps between unit disks in \( \mathbb{C}^1 \). The simplest example is \( g: \Delta = \{z \in \mathbb{C}^1 \mid |z| < 1\} \to \Delta, g(z) = z^n \), where \( n \) is any positive integer. Nevertheless, such a phenomenon is no longer true in higher-dimensional cases. H. Alexander was able to verify the following interesting fact.

**Theorem 1** [1]. Let \( B_n = \{(z_1, z_2, \ldots, z_n) \mid \sum_{i=1}^{n} |z_i|^2 < 1\} \) be the unit ball in \( \mathbb{C}^n \), \( n \geq 2 \). Suppose \( f: B_n \to B_n \) is a proper holomorphic mapping. Then \( f \) must be a biholomorphism.

The following result due to S. Pincuk is an extension of Alexander's theorem.

**Theorem 2** [5]. Let \( D_1 \) and \( D_2 \) be two strongly pseudoconvex bounded domains with smooth boundaries in \( \mathbb{C}^n \), \( n \geq 2 \). Suppose \( f: D_1 \to D_2 \) is a proper holomorphic mapping. Then \( f \) is a covering.

In [7] the second author proved the following result concerning biholomorphic groups of strongly pseudoconvex domains.

**Theorem 3** [7]. Let \( D \) be a strongly pseudoconvex bounded domain with smooth boundary in \( \mathbb{C}^n \). Then \( \text{Aut}(D) \) is noncompact iff \( D \) is biholomorphic to \( B_n, n = \text{dim} \, D \).
In view of a lot of recent attention on the topic of proper holomorphic mappings, the authors feel that it might be worthwhile to point out the following startling fact which generalizes Theorem 3.

**Theorem 4.** Let $D_1$ and $D_2$ be two strongly pseudoconvex bounded domains with smooth boundaries in $\mathbb{C}^n$, $n \geq 2$. Then $P(D_1, D_2)$ is noncompact iff both $D_1$ and $D_2$ are biholomorphic to $B_n$, where $P(D_1, D_2)$ denotes the set of all proper holomorphic mappings between $D_1$ and $D_2$.

Pincuk's Theorem 2 says that proper holomorphic mappings between strongly pseudoconvex domains are unbranching. It follows that Theorem 4 is an immediate consequence of the local version stated next, which is the principal result of this note.

**Theorem 5.** Let $D_1$ and $D_2$ be bounded domains in $\mathbb{C}^n$. We denote $P_0(D_1, D_2)$ as the set of all unbranching proper holomorphic maps from $D_1$ to $D_2$. Suppose the following two conditions are fulfilled.

1. There is a strongly pseudoconvex boundary point $p \in \partial D_2$.
2. There exists a point $x \in D_1$ and a sequence $\{f_j\} \subseteq P_0(D_1, D_2)$ such that $\{f_j(x)\}$ converges to $p$.

Then both $D_1$ and $D_2$ are biholomorphic to $B_n$.

(A) **Some preliminaries and related results.** Let $M$ be a complex manifold of dimension $n$, $x \in M$, and $k$ an integer between one and $n$.

**Definition.** The Eisenman differential $k$-measure on $M$ is a function $E^k_M$:

$$E^k_M(x, v) = \inf \left\{ R^{-2k} | \text{there exists a holomorphic map } f : B_k(R) \rightarrow M \text{ such that } f(0) = x \text{ and } \left( \frac{\partial}{\partial w_1} \wedge \frac{\partial}{\partial w_2} \wedge \cdots \wedge \frac{\partial}{\partial w_k}(0) \right) = v \right\},$$

where $B_k(R) = \{ w = (w_1, w_2, \ldots, w_k) \in C^k \mid \sum_{i=1}^k |w_i|^2 < R \}$. When $k = 1$, it is called a Kobayashi-Royden differential metric \cite{6}, denoted $K_M = k\sqrt{E^1_M}$. As $k = n$, it is a volume form, denoted by $E^n_M = |E^n_M| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$, where $|E^n_M|$ is a function on $M$.

On the other hand, the Carathéodory differential $k$-measure $C^k_M$ is defined as follows.

**Definition.** $C^k_M : \wedge^k T_x(M) \rightarrow \mathbb{R}$, $(x, v) \in \wedge^k T_x(M)$, $C^k_M(x, v) = \sup \{ 1/R^{2k} \mid \text{there exists a holomorphic mapping } f : M \rightarrow B_k(R) \text{ such that } f(x) = 0, df_x(v) = \partial/\partial w_1 \wedge \cdots \wedge \partial/\partial w_k(0) \}$.

When $k = 1$, it is called a Carathéodory-Reiffen differential metric, denoted by $C_M = \sqrt{C^1_M}$. As $k = n$, it is a volume form $C^n_M = |C^n_M| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$, where $|C^n_M|$ is a function on $M$.

One can also define $E^k_M$ and $C^k_M$ relative to a polydisc instead of a ball. They are different measures, but enjoy similar properties. In the sequel, we shall use $I^k_M$ to represent either $E^k_M$ or $C^k_M$.

The following theorem follows almost immediately from the definitions \cite{4}.
THEOREM (a). (1) $E_M^k \geq C_M^k$ on any complex manifold $M$.

(2) Let $f: M_1 \rightarrow M_2$ be a holomorphic mapping between complex manifolds $M_1$ and $M_2$. Then one has $I_{M_1}^k \geq f^*(I_{M_2}^k)$, a measure-decreasing property under $f$.

(3) Let $X$ be a domain of a complex manifold $Y$. Then $I_X^k \geq I_Y^k$, a monotone property, holds.

(4) Any biholomorphism $f$ of a complex manifold $X$ is measure-preserving relative to $I_X^k$, that is, $I_X^k = f^*(I_X^k)$.

(5) Let $\widetilde{M}$ be a covering of a complex manifold $M$. Denote $\pi: \widetilde{M} \rightarrow M$ as the covering projection. Then $E_{\widetilde{M}}^k = \pi^*(E_M^k)$.

THEOREM (b) [3, 8]. Let $D$ be a bounded domain in $\mathbb{C}^n$ with a strongly pseudoconvex boundary point $p \in \partial D$. We denote $\widetilde{D} = V \cap D$, where $p \in V$ is a sufficiently small ball in $\mathbb{C}^n$. Then the following is true: $|E_D^k(z)/C_D^k(z)|$ approaches one as $z \rightarrow p$.

In [7], the next theorem was proved for the special case where $D$ is completely hyperbolic. Actually, a similar proof can yield a slightly more general statement as follows.

THEOREM (c) [7]. Let $D$ be a bounded domain in $\mathbb{C}^n$. Suppose that there is one point $x \in D$ such that $|E_D^k(x)| = |C_D^k(x)|$. Then $D$ is biholomorphic to the euclidean ball.

THEOREM (d) [2] (CARTAN'S FIXED POINT THEOREM). Let $(X, ds^2)$ be a simply-connected complete Riemannian manifold with nonpositive sectional curvature. Suppose $G$ is a compact Lie group acting on $X$ as isometries. Then $G$ has a fixed point.

In particular, any finite group $H$ acting on $X$ isometrically must fix at least one point.

THEOREM (e). Let $D_1$ and $D_2$ be bounded domains in $\mathbb{C}^n$. Suppose that

(1) there is a strongly pseudoconvex point $p \in \partial D_2$;

(2) one can find $x \in D_1$ and a sequence of holomorphic mappings $\{f_j\} \subset \text{Hol}(D_1, D_2)$ such that $\{f_j(x)\} \rightarrow p$.

Then there exists a subsequence of $\{f_j\}$, denoted by the same notation $\{f_j\}$, satisfying the property: For any compact set $K \subset D_1$ and any open set $\widetilde{D} = V \cap D_2$, where $p \in V$ is an open set in $\mathbb{C}^n$, there is a $j_0$ in such a way that $f_j(K) \subset \widetilde{D}$ for all $j \geq j_0$.

PROOF. Since $\{f_j(x)\} \rightarrow p$, by normal family argument one can find a subsequence of $\{f_j\}$ converging on compacta to a holomorphic mapping $f: D_1 \rightarrow \mathbb{C}^n$ so that $f(z) = p$ and $f(D_1) \subset \partial D_2$. By assumption, $\partial D_2$ is strongly pseudoconvex at $p$ and it contains no complex analytic variety of positive dimension through $p$. This implies $f$ is a constant mapping which brings the whole $D$ onto a single point. Our claim in Theorem (e) should now be clear.

(B) Proof. Let us assume $|E_{D_1}^n(x)| = |C_{D_1}^n(x)|$ for the given point $x$ in $D_1$. By Theorem (c), this implies that $D_1$ must be biholomorphic to $B_n$. If the order of the covering $f_j: B_n = D_1 \rightarrow D_2$ is greater than one, this would contradict Cartan’s fixed point theorem (Theorem (c)) because the Bergman metric on $B_n$ has
negative sectional curvature and it is invariant under biholomorphisms. Thus $D_2$ is also biholomorphic to $B_n$. Therefore, the whole proof depends on the following assertion.

Claim. $|E^n_{D_1}(x)| = |C^n_{B_1}(x)|$.

Proof. For each $j$, $f_j : D_1 \to D_2$ is a covering. From Theorem (a)(5) we have
$$E^n_{D_1}(x, v) = E^n_{D_2}(x_j, df_j(v)),$$
where $x_j = f_j(x)$ and $(x, v)$ is a nonzero $n$-vector at $x$. Let $(D_1)_k$ be an increasing sequence of domains such that $\bigcup_{k=1}^\infty (D_1)_k = D_1$, $x \in (D_1)_k$ for each $k$, and $(D_1)_k \subset (D_1)_{k+1}$. For each $j$, denote $(D_2)_j^k = f_j(D_1)_k$. For a fixed $k$, we obtain by Theorem (a)(2)(3) the inequalities
$$C^n_{(D_1)_k}(x, v) \geq C^n_{(D_1)_j^k}(x_j, df_j(v)) \geq C^n_D(x_j, df_j(v)).$$
The last inequality on the above chain is valid for sufficiently large $j$. The reason is that when $j$ is sufficiently large, $f_j((D_1)_k) = (D_2)_j^k \subset \tilde{D}$ by Theorem (e), where $\tilde{D} = V \cap D_2$, $p \in V$ is an open set in $\mathbb{C}^n$. It follows that for fixed $k$ and large $j$, we have the chain
$$C^n_{(D_1)_k}(x, v) \geq C^n_{(D_2)_j^k}(x_j, df_j(v)) \geq C^n_D(x_j, df_j(v))$$
of inequalities (Theorem (a)(5) has been used here).

Observe that:

(i) By the volume decreasing property under holomorphic mappings (Theorem (a)(2)), we have $E^n_{D}(x_j, df_j(v)) \geq E^n_{D_2}(x_j, df_j(v))$ as the inclusion map $\tilde{D} \hookrightarrow D_2$ is holomorphic. Therefore, we have
$$\frac{C^n_{(D_1)_k}(x, v)}{E^n_{D_1}(x, v)} \geq \frac{C^n_{(D_2)_j^k}(x_j, df_j(v))}{E^n_{D_2}(x_j, df_j(v))}.$$

(ii) Again by the strong pseudoconvexity of $p \in \partial D_2$, one obtains
$$\frac{C^n_{D}(x_j, df_j(v))}{E^n_{D}(x_j, df_j(v))} \to 1 \quad \text{as} \quad x_j \to p$$
by Theorem (b).

(iii) If we let $k \to \infty$, then $C^n_{(D_1)_k}(x, v) \to C^n_{(D_1)}(x, v)$. This approximation property can be proved by elementary normal family argument.

(iv) It is always true that $C^n_{(D_1)}(x, v)/E^n_{(D_1)}(x, v) \leq 1$ by Theorem (a)(1).

Combining (i)–(iv), and letting $j \to \infty$ and then $k \to \infty$, one concludes that $1 \geq C^n_{(D_1)}(x, v)/E^n_{(D_1)}(x, v) \geq 1$, proving our claim.

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