JONES' BRAID-PLAT FORMULA AND A NEW SURGERY TRIPLE

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ABSTRACT. A link $L_\beta(2k, n - 2k)$ is defined by a type $(2k, n - 2k)$ pairing of an n-braid β if the first $2k$ strands are joined up as in a plat and the remaining $n - 2k$ as in a closed braid. The main result is a formula for the Jones polynomials of $L_\beta(2k, n - 2k)$, valid for all k, $0 \leqslant 2k \leqslant n$, which generalizes and relates earlier results of Jones for the cases $n = 0$ and $2k$.

1. Introduction. Let B_n denote Artin's n-string braid group [Ar], $n = 1, 2, 3, \ldots$. Thinking of elements of B_n as geometric braids, there are two well-known ways, dating back to Alexander [Al] and Reidemeister [R], to construct a link from an element α of B_n. The first identifies the n free ends at the beginning of α with the n free ends at the end of α to form an (oriented) *closed braid*, denoted $\hat{\alpha}$. The second, assuming n to be even, identifies adjacent pairs of strands at each end of α to form an (unoriented) *plat* $\check{\alpha}$.

Let A_n, $n = 1, 2, \ldots$, be the sequence of von Neumann algebras described in [J1] generated by projections $e_1, \ldots, e_{n-1}$. Let $r_t\colon B_n \to A_n$, $t \in \mathbf{C}$, be the 1-parameter family of representations of B_n in A_n, and let tr: $A_n \to \mathbf{C}$ be the Jones trace. For each $\alpha \in B_n$, Jones defines a polynomial

$$(1) \qquad V_{\hat{\alpha}}(t) = \delta^{n-1} \operatorname{tr}(r_t(\beta)),$$

where $\delta = -(1 + t)/\sqrt{t}$, and shows that it is an invariant of the oriented link type of $\hat{\alpha}$.

Starting with a fixed braid $\alpha \in B_{2k}$ it seems on the face of it that the link types of $\check{\alpha}$ and $\hat{\alpha}$ are quite unrelated; for example, one may alter α so that $\check{\alpha}$ remains fixed, while $\hat{\alpha}$ is changed from a knot to a link. It therefore seemed remarkable when Jones discovered in early 1985 (see [J2]) that their polynomial invariants were given by closely related formulas, viz (1) and

$$(2) \qquad V_{\check{\alpha}}(t) \equiv \delta^{3k-1} \operatorname{tr}(r_t(\alpha) e_1 e_3 \cdots e_{2k-1}),$$

where $\equiv$ means up to a multiplicative power of $\sqrt{t}$. Note that the discovery of (2) as reported in [**J2**] was inspired by work on the Potts model in statistical mechanics, and so involved a chain of ideas which seemed very far from the studies which led to the discovery of (1).

One can also define a link from a closed braid by joining up the $2n$ free ends of the braid strings in pairs in still other ways. Assuming that the braid is defined by a regular projection onto a plane P, we restrict our attention to pairings in which the free ends of α are joined up by n disjoint arcs in P (since if the arcs were to cross, the crossings could be subsumed into the braid). Call such a pairing *type* $(2k, n - 2k)$ if the first k pairs of strings at both ends of the braid are joined up as in a plat and the remaining $n - 2k$ as in a closed braid (see Figure 1a) and let $L_\alpha(2k, n - 2k)$ denote the link so obtained. Up to canonical modifications in the defining braid, all noncrossed pairings may be obtained in this way.
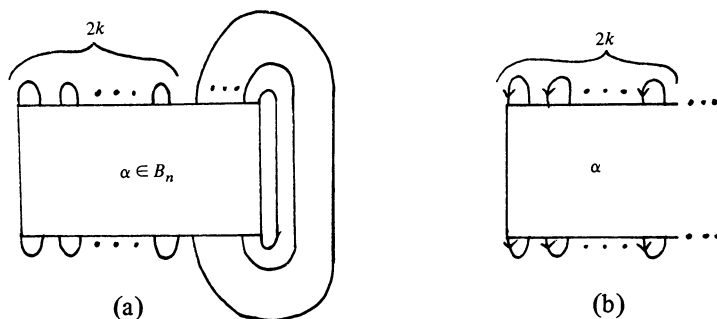

FIGURE 1

The first result in this paper (Theorem 1) generalizes formulas (1) and (2) to a single formula for the Jones polynomial of $L_\alpha(2k, n - 2k)$, $0 \leqslant 2k \leqslant n$. At the same time we will show that if one places a mild restriction on the defining braid α, then there is a well-defined "standard" way to orient the plat $L_\alpha(2k, n - 2k)$, and with this choice of orientation we can identify the multiplicative power of t in (2) precisely. Our proof is elementary and different from that in [**J2**], and it explains the relationship between (1) and (2).

The second result is an application of Theorem 1 to prove that there is a "surgery triple" which relates the polynomials V_{L_1}, $V_{L_{-1}}$, V_{L_∞} of three links L_1, L_{-1}, L_∞. Let L_0, L_1, L_{-1}, L_∞ be four links which are defined by diagrams (no longer necessarily coming from braids) which are identical everywhere outside a small disc D, and are as illustrated in Figure 2 in D. Note that L_0, L_1, L_{-1} are all oriented, but the component of L_∞ which is associated to the strands inside D has no natural
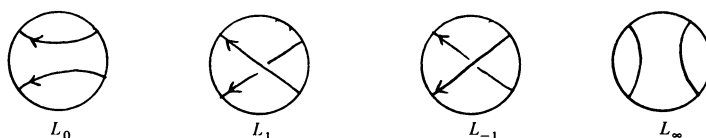

FIGURE 2

orientation. By Theorem 12 of [J1] the Jones polynomials of L_0, L_1, L_{-1} are related by

$$(3) \qquad\qquad V_{L_0} = \frac{1}{\mu}\left(t^{-1}V_{L_1} - tV_{L_{-1}}\right),$$

where $\mu = (t - 1)/\sqrt{t}$. (Caution: see Comment 4.2 at the end of this paper.) We will prove (Theorem 2) that there is a similar formula which relates V_{L_∞}, V_{L_1}, and $V_{L_{-1}}$. The most interesting case (the Corollary) occurs when L_1, and hence also L_{-1} and L_∞, are knots.

REMARKS. This manuscript subsumes two earlier versions, [B] and [K]. Our results in those two papers were related, with [B] proving a more general result than [K], but by a more complicated method than that in [K]. We decided to combine our two papers.

Theorem 2 was originally proved simultaneously by Jones (private letter) and by the first author in [B]. It was then used by Lickorish in [L] where a different proof is given. The proof which we give here is different from both that in [B] and that in [L].

2. The Jones polynomial of $L_\alpha(2k, n - 2k)$.

The key to pinning down the multiplicative power of t in (2) and in its generalization (below) will be seen to be related to having control over orientations in $L_\alpha(2k, n - 2k)$. With this in mind we begin by placing a (mild) restriction on the defining braid $\alpha \in B_n$. The braid α will be said to be *admissible* for $L_\alpha(2k, n - 2k)$ if the partial orientations on $L_\alpha(2k, n - 2k)$ are those indicated in Figure 1b.

LEMMA 1. *Any braid* $\beta \in B_n$ *may be altered to an admissible braid* α, *with* $L_\alpha(2k, n - 2k) = L_\beta(2k, n - 2k)$, *by adding appropriate half-twists at the top and bottom of* β.

PROOF. Clear. □

Assume from now on that $\alpha \in B_n$ is admissible for $L_\alpha(2k, n - 2k)$, and that the plat part has been oriented as in Figure 1b. If this does not determine orientations on all components of $L_\alpha(2k, n - 2k)$, then the unoriented components may be oriented as in a closed braid, say from top to bottom in the braid part. We call this the *standard orientation* for $L_\alpha(2k, n - 2k)$. Observe that the standard orientation on $L_\alpha(2k, n - 2k)$ determines orientations on the individual braid strands of α. When we need to specify these, we will write $\bar{\alpha}$, to distinguish $\bar{\alpha}$ from $\vec{\alpha}$, the same braid with each strand oriented top-to-bottom.

LEMMA 2. *Let* $\bar{\alpha}$ *be defined as above. If the strands of* $\bar{\alpha}$ *are identified by a type* $(0, n)$ *pairing, as in a closed braid, then the resulting link* $\bar{L}_\alpha(0, n)$ *can be given a consistent orientation, using the orientations on the strands of* $\bar{\alpha}$.

PROOF. Modify the diagram for $L_\alpha(2k, n - 2k)$ by pulling the upper plat parts around the braid until they are next to the corresponding lower plat parts, as in Figure 3. Now do "surgery" on the link, changing the tangles inside the dotted discs from the tangle labeled 0 in Figure 2 to the tangle labeled 1. The result will be $\bar{L}_\alpha(0, n)$ oriented consistently to agree with the orientation on $\bar{\alpha}$. □
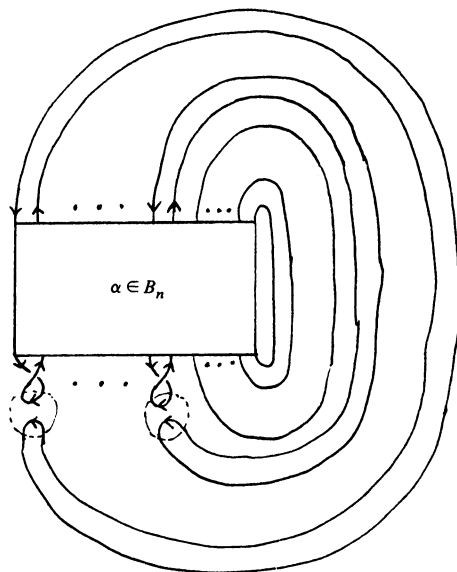
FIGURE 3

We are now ready to state and prove our first result. Let $\vec{L}_\alpha(0, n)$ be the closed braid link determined by $\vec{\alpha}$. Then $\overleftarrow{L}_\alpha(0, n)$ is obtained from $\vec{L}_\alpha(0, n)$ by reversing the orientation of the "wrongly ordered" sublink $\overleftarrow{K}$. Let λ be the linking number $\text{lk}(\overleftarrow{K}, \vec{L}_\alpha(0, n) - \overleftarrow{K})$.

THEOREM 1. *Let $\alpha \in B_n$ be admissible for $L_\alpha(2k, n - 2k)$. Assume that the link $L_\alpha(2k, n - 2k)$ has the standard orientation. Then*

$$(4) \qquad V_{L_\alpha(2k, n-2k)} = t^{3\lambda} \delta^{n+k-1} \text{tr}\left(r_t(\alpha) e_1 e_3 \cdots e_{2k-1} \right).$$

PROOF. See Figure 3. Let $\{ L_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}; \; \varepsilon_i \in \{0, 1, -1\} \}$ be the collection of 3^k links which are obtained from the link in Figure 3 by inserting in each of the k discs the tangles from Figure 2 which correspond to the choices of $\varepsilon_1, \ldots, \varepsilon_K$. Note that each triplet $(L_{\varepsilon_1 \cdots 0 \cdots \varepsilon_K}, L_{\varepsilon_1 \cdots 1 \cdots \varepsilon_K}, L_{\varepsilon_1 \cdots -1 \cdots \varepsilon_K})$ is a $0, 1, -1$ surgery triple, so that the corresponding Jones polynomials are related by equation (3).

Consider the special cases when each $\varepsilon_i = +1$. Examples are given in Figure 4, when $2k = n = 4$. Each $\overleftarrow{L}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}$ is the closure of a braid $\vec{\alpha}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K} = \alpha \sigma_1^{1-\varepsilon_1} \sigma_3^{1-\varepsilon_2} \cdots \sigma_{2K-1}^{1-\varepsilon_K}$, and so can be obtained from the closed braid $\vec{L}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}$ by reversing the orientation on a sublink $\vec{K}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}$. We study how the linking numbers $\lambda_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K} = \text{lk}(\vec{K}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}, \vec{L}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K} - \vec{K}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K})$ vary for different choices of the ε_i's in the set $\{ \pm 1 \}$. Note that each appearance of -1 in the array $\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K$ corresponds to a square σ_{2i-1}^2 in $\alpha \sigma_1^{1-\varepsilon_1} \sigma_3^{1-\varepsilon_2} \cdots \sigma_{2K-1}^{1-\varepsilon_K}$. Now, the portion of the link diagram which is associated to this square always involves distinct components. The reason is that the two strands are oppositely oriented, and there is no way for an upward-oriented braid strand to be joined to a downward-oriented braid strand in a link of type $(0, n)$. Let $m_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}$ be the number of occurrences of -1 in the array $\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K$.

Then

(5)
$$\lambda_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K} = -\lambda + m_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}.$$

Now Jones' reversing result (see [L-M] or [M]) tells us that

(6)
$$\overleftarrow{V}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K} = t^{-3\lambda_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}} \to \overrightarrow{V}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}.$$

Formulas (1), (5), and (6) then imply that the polynomial $\overleftarrow{V}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}$ is given by

(7)
$$\overleftarrow{V}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K} = \delta^{n-1} t^{3\lambda - 3m_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}} \operatorname{tr}\left(a g_1^{1-\varepsilon_1} g_3^{1-\varepsilon_2} \cdots g_{2K-1}^{1-\varepsilon_K} \right)$$

where $a = r_t(\alpha)$, $g_i = r_t(\sigma_i) = \sqrt{t}\,(te_i - (1 - e_i))$.

We will have no more use for $\overleftarrow{L}_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}$, so from now on we drop the arrow, writing α, L, V for $\bar{\alpha}$, $\bar{L}$, $\bar{V}$. We are ready to compute $V_{00 \cdots 0} = V_{L_\alpha(2k, n-2k)}(t)$. The reader may find it helpful to follow the pictures in Figure 4, which illustrate the case $2k = n = 4$. Construct a tree of $2^{k+1} - 1$ links, arranged in $k + 1$ rows, with $L_{00 \cdots 0}$ at the base. Place the two links $\{L_{\varepsilon_1 0 \cdots 0}; \ \varepsilon_1 = \pm 1\}$ in the second row from the bottom, the four links $\{L_{\varepsilon_1 \varepsilon_2 0 \cdots 0}; \ \varepsilon_i = \pm 1\}$ above these, and so forth, ending with the 2^k links $\{L_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_K}; \ \varepsilon_i = \pm 1\}$ in the top row. There is a natural order in each row, dictated by working up from the bottom and requiring that the links in the tree group themselves into $(0, +, -)$ surgery triples $(L_{\varepsilon_1 \cdots 0 \cdots \varepsilon_K}, L_{\varepsilon_1 \cdots 1 \cdots \varepsilon_K}, L_{\varepsilon_1 \cdots -1 \cdots \varepsilon_K})$, with the "$+$" and "$-$" member of each triple directly above the "0" member. Applying (3) repeatedly, we can then work our way up the tree, to express $V_{00 \cdots 0}$ as a sum of the polynomials of the links which lie above $L_{00 \cdots 0}$ in the tree, viz:

$$V_{00 \cdots 0} = \frac{1}{\mu}\left(t^{-1} V_{10 \cdots 0} - t V_{-10 \cdots 0} \right)$$

$$= \frac{1}{\mu^2}\left(t^{-1}\left(t^{-1} V_{110 \cdots 0} - t V_{1-10 \cdots 0} \right) - t\left(t^{-1} V_{-110 \cdots 0} - t V_{-1-10 \cdots 0} \right) \right)$$

$$= \cdots = \frac{1}{\mu^k} \sum_{\varepsilon_i = \pm 1} (-1)^{m_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k}} t^{-k - 2m_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k}} V_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k}$$

which by virtue of (7) becomes

$$V_{00 \cdots 0} = \frac{\delta^{n-1} t^{3\lambda}}{\mu^k} \sum_{\varepsilon_i = \pm 1} (-1)^{m_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k}} t^{-k - m_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k}} \operatorname{tr}\left(a g_1^{1-\varepsilon_1} g_3^{1-\varepsilon_2} \cdots g_{2k-1}^{1-\varepsilon_k} \right).$$

Magically, one recognizes that the 2^k terms in the sum combine into a single formula:

(8)
$$V_{00 \cdots 0} = \frac{\delta^{n-1} t^{3\lambda} (-1)^k t^{-2k}}{\mu^k} \operatorname{tr}\left(a \left(g_1^2 - t \right)\left(g_3^2 - t \right) \cdots \left(g_{2k-1}^2 - t \right) \right).$$

Then $g_i = \sqrt{t}\,((t + 1)e_i - 1)$ (see [J1]) implies that

(9)
$$g_i^2 - t = -t^2 \mu \delta e_i, \qquad 1 \leqslant i \leqslant n - 1.$$

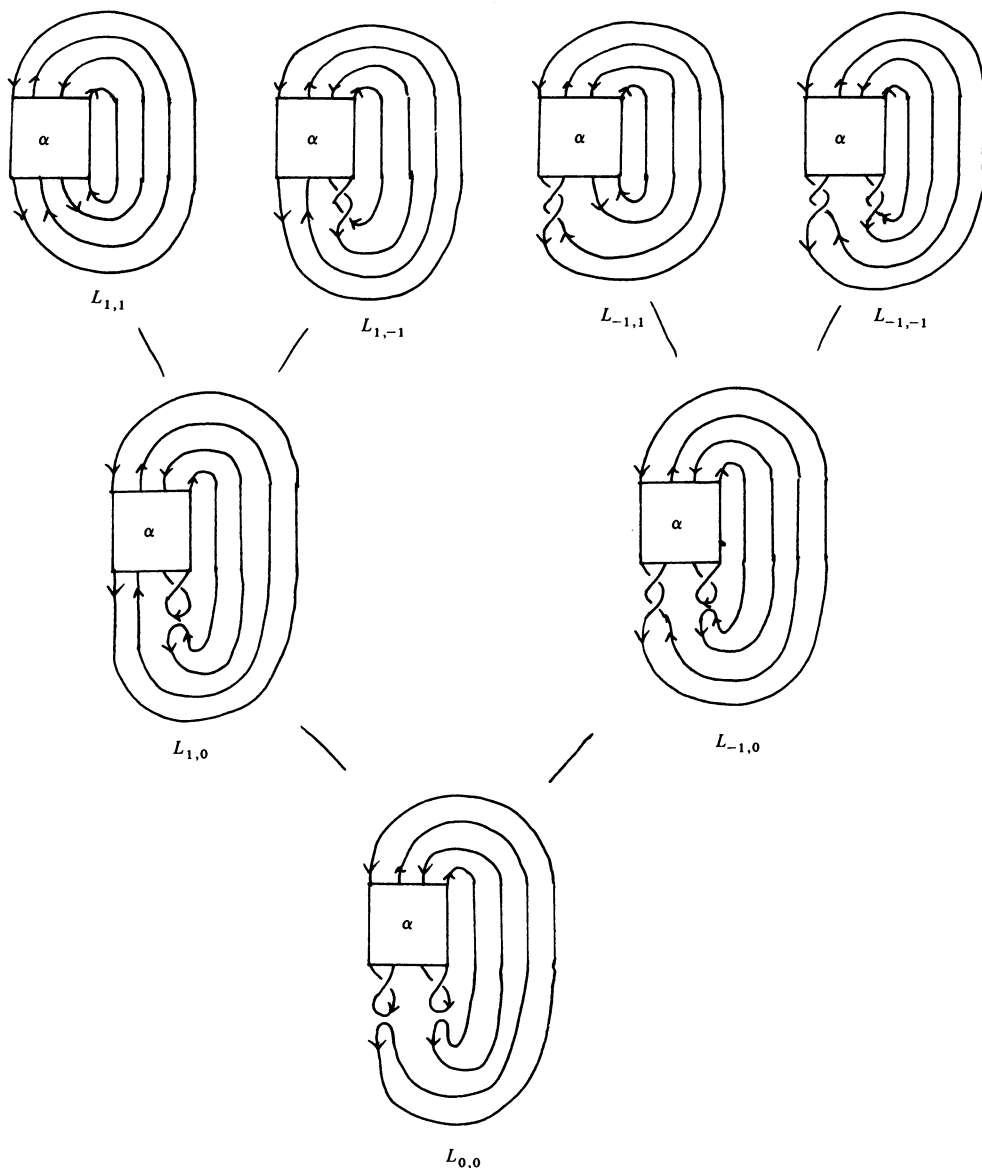Substituting (9) into (8) we obtain (4), and our proof is complete. $\qquad \square$

FIGURE 4

3. Surgery triples. In this section we apply Theorem 1 to give the promised formula which relates the polynomials V_1, V_{-1}, V_∞ of three links L_1, L_{-1}, L_∞ which are defined by the surgeries in Figure 2.

Recall that if K is a sublink of L, then $\lambda(K)$ means $\mathrm{lk}(K, L - K)$. Note that there is no canonical way to orient the components of L_∞ which extend the surgered arcs. The reader is referred to Figure 5 for the description of the various possible components K_j^i of L_j, $j = 1, -1, \infty$.
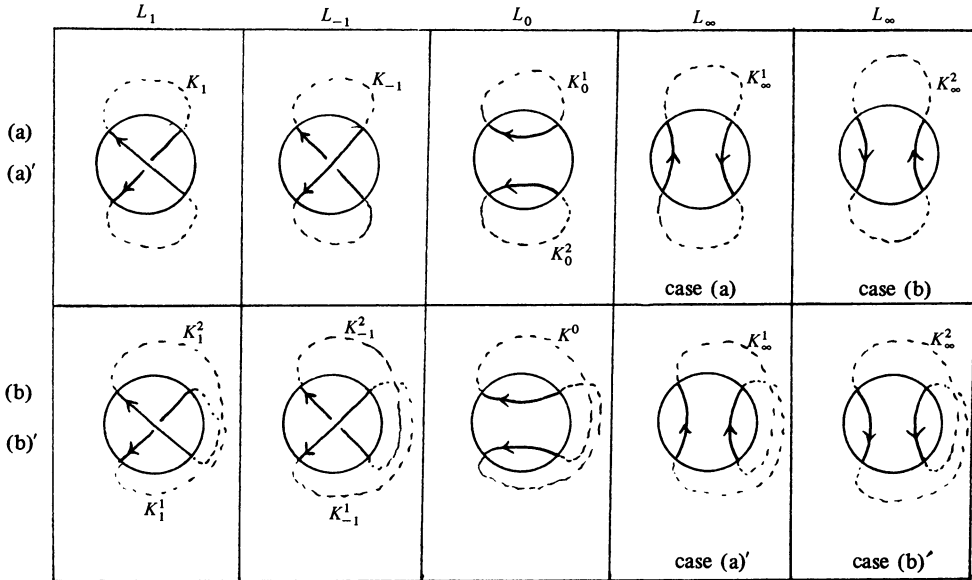
FIGURE 5

THEOREM 2. *The Jones polynomials of L_1, L_{-1}, L_∞ are related by*

(10)
$$V_1 - tV_{-1} + t^{3q}(t-1)V_\infty = 0$$

where q is given by

case	q
a	$\lambda(K_0^2)$
b	$\lambda(K_{-1}^1) + \frac{1}{2}$
a'	$\lambda(K_0^2) - \lambda(K_\infty^2)$
b'	$\lambda(K_{-1}^1) + \frac{1}{2} - \lambda(K_\infty^2)$

PROOF. We consider the cases (a) and (b), since (a') and (b') follow from (a) and (b) by Jones' reversing result. For the purposes of the proof, we may assume that L_1 is a closed braid, for any link may be so represented, and the deformations which take it to a closed braid can be taken to avoid the surgered disc. By isotopy and conjugation, the crossing of interest can then be moved to the lower left part of the braid. Finally, inserting $\sigma_1^{-1}\sigma_1$ in the braid, if necessary, we can assume the braid has the form $\beta\sigma_1^2$. Thus we assume: $L_1 = \widehat{(\beta\sigma_1^2)}$, $L_{-1} = \hat{\beta}$, $L_\infty = L_{\beta\sigma_1}(2, n-2)$ for (a) and $L_\beta(2, n-2)$ for (b). By Theorem 1 we then have

(11)
$$V_\infty = \begin{cases} t^{3\lambda}\delta^n \operatorname{tr}(r_t(\beta\sigma_1)e_1) & \text{for (a)}, \\ t^{3\lambda}\delta^n \operatorname{tr}(r_t(\beta)e_1) & \text{for (b)}. \end{cases}$$

The integer λ in equation (11) is defined as in Theorem 1, viz: orient the plat part of L_∞ as in Figure 1b, and then construct the associated closed braid (in our case it will be $\widehat{(\beta\sigma_1)}$ or $(\hat{\beta})$); then λ is the linking number of the "wrongly ordered" sublink of that closed braid.

Apply (9) and (1) to (11). For (b), this gives

$$V_\infty = -t^{3\lambda-2}\mu^{-1}\delta^{n-1}\operatorname{tr}\left[r_t(\beta)\left(g_1^2 - t\right)\right]$$

$$= -t^{3\lambda-2}\mu^{-1}[V_1 - tV_{-1}],$$

$$V_1 - tV_{-1} + t^{-3(\lambda-1/2)}(t-1)V_\infty = 0.$$

For (a), noting $g_1 e_1 = t^{3/2}e_1$, we have

$$V_1 - tV_{-1} + t^{-3\lambda}(t-1)V_\infty = 0.$$

It is easy to check that $\lambda = -\lambda(K_0^2)$ for (a) and $-\lambda(K_{-1}^1)$ for (b). □

COROLLARY. *If L_1 is a knot, then L_{-1} and L_∞ are likewise, and*

(12) $$V_1 - tV_{-1} + t^{3\lambda(L_0)}(t-1)V_\infty = 0,$$

where $\lambda(L_0)$ is the total linking number of L_0.

PROOF. The assertions about connectivity are proved in Figure 5. If L_1 is a knot, then L_0 has two components, and the result follows from Theorem 2. □

4. Comments.

4.1. The Corollary allows one to compute $V_k(t)$ for an arbitrary knot k by choosing a minimal set of crossing changes which unknot k, and then constructing a tree of knots in which each interior node corresponds to a single $(1, -1, \infty)$ surgery and each braid ends in the unknot, with polynomial 1. Orientations will not matter, because we are working with knots. In principle, this ought to give a very simple tool for investigating $V_k(t)$ for knots, were it not for the fact that each time one does the $(1, -1, \infty)$ surgery one has to compute $\lambda(L_0)$, and the numbers so obtained seem unpredictable. They appear to play a crucial role in $V_k(t)$.

4.2. In this paper we use conventions in our definition of V_L which are consistent with formula (3), correcting an error in Theorem 12 of [J1], which is inconsistent with earlier conventions in that same paper.

4.3. Very new results of Louis Kauffman (private letter) show that e_i's in Jones' algebra A_n may be interpreted geometrically as braid-like objects (see Figure 6) which compose in much the same way as braids, by concatenation and rescaling. With this interpretation the formula (2) becomes transparent, because on adding the "weaving pattern" $e_1 e_3 \cdots e_{2k-1}$ to the end of a geometric braid α and then identifying top and bottom as in a closed braid, one obtains immediately the link $L_\alpha(2k, n - 2k)$.
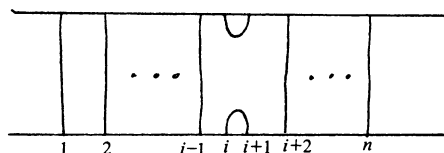

FIGURE 6

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