ON THE CARDINALITY OF A TOPOLOGY

RUAN YONGBIN

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ABSTRACT. Let \( o(X) \) denote the cardinality of topology of a space \( X \). I. Juhasz proves that \( o(X)^\omega = o(X) \) for regular hereditarily paracompact spaces. We prove it for more general classes of spaces.

If \( X \) is a topological space, let \( o(X) \) denote the cardinality of its topology. A number of papers \([1, 2, 3]\) have investigated classes for which \( o(X) = o(X)^\omega \). We generalize several of these results by proving

**Theorem 1.** For every infinite hereditarily weakly \( \theta \)-refinable \( T_2 \) space \( X \), \( o(X) = o(X)^\omega \).

A space \( X \) is weakly \( \theta \)-refinable \([4]\) if and only if each open cover \( \mathcal{U} \) of \( X \) has a refinement \( \mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n \) where each \( \mathcal{V}_n \) is a discrete family of subsets of the space \( \bigcup \mathcal{V}_n \).

If \( \eta \) and \( \mu \) are infinite cardinals and \( \mu' < \mu \) implies \( \eta^{\mu'} < \eta^\mu \) and \( \eta' < \eta \) implies \( \eta^{\eta'} < \eta^\eta \), then \( \eta^\mu \) is called a jump. By Theorem 0.2 of \([2]\), if \( \eta^\mu \) is a jump, \( \mu \) is regular.

Since \( X \) is \( T_3 \), by Theorem 4.3 of \([2]\), if \( \mu \) is an uncountable cardinal of cofinality \( \omega \) and there is a discrete subset \( S' \) of \( X \) with \( |S'| = \mu' \) for all \( \mu' < \mu \), then there is a discrete subset \( S \) of \( X \) with \( |S| = \mu \).

Before proving the theorem, we also need

**Lemma.** If \( \langle \kappa_\alpha | \alpha \in \gamma \rangle \) is a sequence of infinite cardinals, then there is an \( S \subset \gamma \) with \( \prod_{\alpha \in S} \kappa_\alpha = (\sup_{\alpha \in S} \kappa_\alpha)^{|S|} \).

**Proof of Lemma.** Enumerate the \( \kappa_\alpha \) so that they form a nondecreasing sequence. Let \( S \subset \gamma \) be of minimal order type so that \( \prod_{\alpha \in S} \kappa_\alpha = \prod_{\alpha \in \gamma} \kappa_\alpha \). Without loss of generality, suppose \( S \) is infinite. Note that if \( \beta < \sup S \), then \( S - (\beta + 1) \) has the order type of \( S \), in particular \( |S - (\beta + 1)| = |S| \). So if \( |S| = \lambda \) we have that \( S = \lambda \cdot \alpha \) for \( \alpha = 1 \) or for some limit ordinal \( \alpha < \lambda^+ \). Split \( S \) into \( \lambda \) many cofinal disjoint sets \( \{ A_\beta | \beta \in \lambda \} \). Note that every \( A_\beta \) is cofinal in \( S \) and so

\[
\prod_{\delta \in A_\beta} \kappa_\delta \supseteq \sup_{\delta \in A_\beta} \kappa_\delta = \sup_{\alpha \in S} \kappa_\alpha = \kappa.
\]
Hence
\[ \kappa^\lambda \leq \prod_{\beta \in \lambda} \prod_{\delta \in \alpha} \kappa_\delta = \prod_{\alpha \in \gamma} \kappa_\alpha; \quad \text{so} \quad \kappa^\lambda = \prod_{\alpha \in \gamma} \kappa_\alpha. \]

**Proof of Theorem.** Assume not and let \( X \) be a counterexample; so \( o(X) = \kappa < \kappa^\omega \). Let \( \lambda = \min\{ \mu \mid \mu^\omega > \kappa \} \). Then \( \text{cf}(\lambda) = \omega \) and \( \lambda \leq \kappa \).

For \( x \in X \), let \( o(x, X) = \min\{ o(U) \mid x \in U \text{ and } U \text{ is open} \} \) and \( \sigma = \sigma(X) = \sup\{ o(x, X) \mid x \in X \} \).

Note that there exists a finite subset \( F \subseteq X \) such that \( o(X - F) < \lambda \). To see this assume the contrary and let \( \langle \lambda_n \mid n \in \omega \rangle \) be a cofinal sequence in \( \lambda \). At stage \( n \) pick \( x_n \in X - \{ x_{k_0} \mid k < n \} \) with \( o(x_n, X) \geq \lambda_n \). Since \( X \) is regular there is a disjoint open family \( \{ U_n \mid n \in \omega \} \) with \( x_n \in U_n \), hence \( o(U_n) \geq \lambda_n \). So \( o(X) \geq \prod_{n \in \omega} o(U_n) = \lambda^\omega > \kappa \), a contradiction.

Thus we can assume that \( o(x) < \lambda \) and that \( o(X) = o(X - F) \) for all finite \( F \subseteq X \) (recall that our original \( X \) was hereditarily weakly \( \theta \)-refinable).

Let \( \mathcal{U} \) be an open cover of \( X \) such that each \( U \in \mathcal{U} \) has \( o(U) \leq \sigma \). Since \( X \) is weakly \( \theta \)-refinable there is a refinement \( \mathcal{Y} = \bigcup_{n \in \omega} \mathcal{Y}_n \) such that each \( \mathcal{Y}_n \) is a closed discrete family and \( \bigcup_{n \in \omega} \mathcal{Y}_n = V_n \). Assume \( \mathcal{Y}_n = \{ V_{a} \mid a \in \beta_n \} \) and \( \beta_n \) is infinite for each \( n \in \omega \). Then \( \kappa = o(X) \leq \prod_{n \in \omega} o(V_n) \). Now we derive a contradiction by showing that \( \prod_{n \in \omega} o(V_n) < \kappa \).

Fixing \( n \in \omega \), we have \( o(V_n) = \prod_{a \in \beta_n} o(V_{a}^n) = \sigma_n^\mu \) for \( \sigma_n = \sup\{ o(V_{a}^n) \mid a \in \beta_n \} \) and \( \mu_n = |\beta_n| \) by the lemma. Since \( o(V_{a}^n) \leq \sigma \) for every \( a \), we have that \( \sigma_n \leq \sigma \). Also \( \sigma_n^\mu_n < \lambda \) since \( o(V_{a}^n) < o(X) = \kappa \) and \( \sigma_n^\mu_n = \sigma_n^\mu < \kappa \) if \( \mu_n > \omega \); otherwise \( \sigma_n^\mu_n < \sigma < \lambda \). Note that \( \prod_{n \in \omega} \sigma_n^\mu_n < \kappa \) we may assume that \( \sigma \leq \sigma_n^\mu_n \) for all \( n \). Then \( \sigma_n^\mu_n = \sigma^\mu_n < \lambda \) (since \( \sigma_n \leq \sigma \leq \sigma_n^\mu_n < \lambda \)). Hence if \( \mu = \sup_{n \in \omega} \mu_n \), \( \prod_{n \in \omega} \sigma_n^\mu_n = \sigma^\mu \). If \( \sigma^\mu < \lambda \), then \( \prod_{n \in \omega} o(V_n) < \kappa \) as desired.

So assume that \( \sigma^\mu \geq \lambda \). Since \( \sigma^\mu < \lambda \), \( \mu \) is uncountable and, since \( \sigma_n^\mu_n < \lambda \) for all \( n \), \( \text{cf}(\mu) = \omega \). If \( \eta = \min\{ \alpha \mid \alpha^\mu \geq \lambda \} \), then \( \eta' < \eta \) implies \( \eta^\mu < \eta^\mu \) and \( \mu' < \mu \) implies \( \eta^\mu < \eta^\mu \). So if \( \eta \) is infinite, \( \eta^\mu \) is a jump and \( \mu \) is regular. This contradicts \( \mu \) being uncountable and of cofinality \( \omega \).

So it remains to show that \( \eta \) is infinite if \( \mu \) is. To show this, for each \( n \in \omega \) and \( \alpha \in \beta_n \) choose \( x_a \in V_{a}^n \) and let \( S_n = \{ x_a \mid \alpha \in \beta_n \} \). Since \( S_n \) is discrete and \( |S_n| = \mu_n \), there is a discrete \( S \subseteq X \) of cardinality \( \mu \). Thus \( 2^\mu \leq o(S) \leq o(X) = \kappa \). Since \( \mu \) is infinite, \( 2^\mu^\omega = 2^\mu < \kappa \) while \( \lambda^\omega > \kappa \), so \( 2^\mu < \lambda \). Since \( 2^\mu = \eta^\mu \) for finite \( \eta \) and \( \eta^\mu \geq \lambda \) by definition, \( \eta \) is infinite.

Our proof gives the stronger

**Theorem 2.** Let \( X \) be regular space. If for any finite subset \( F \), \( X - F \) is weakly \( \theta \)-refinable, then \( o(X) = o(X)^\omega \).

Since weak \( \theta \)-refinability is hereditary to \( F \)-sets, \( o(X) = o(X)^\omega \) also holds for 1st countable, \( T_3 \), weakly \( \theta \)-refinable spaces or 1st countable, \( T_3 \) paracompact spaces.

A similar proof also yields a theorem, independently proved by I. Juhasz, that \( o(X) = o(X)^\omega \) for all hereditarily weakly collectionwise Hausdorff spaces.
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REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

Current address: Department of Mathematics, University of California-Berkeley, Berkeley, California 94720