A HILBERT CUBE L-S CATEGORY
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ABSTRACT. Let $M$ be a compact connected Hilbert cube manifold ($Q$-manifold). Define $C_z(M)$ to be the smallest integer $k$ such that $M$ can be covered with $k$ open subsets each of which is homeomorphic to $Q \times [0,1)$. Recently L. Montejano proved that, for every compact connected polyhedron $P$, $C_z(P \times Q) = \text{cat}(P) + 1$, where $\text{cat}(P)$ is the Lusternik-Schnirelmann category of $P$. Using a different approach, we prove a noncompact analog of the above theorem by showing that $C_z(P \times Q \times [0,1)) = \text{cat}(P)$ for every compact connected polyhedron $P$.

1. Introduction. For a compact polyhedron $P$, the Lusternik-Schnirelmann (L-S) category of $P$, $\text{cat}(P)$, is the smallest integer $k$ such that $P$ can be covered with $k$ subpolyhedra each of which is null homotopic in $P$. A $Q$-manifold is a separable metric space modeled on the Hilbert cube $Q = [0,1]^\infty$. Let $M$ be a connected $Q$-manifold. Define $C_z(M)$ to be the smallest integer $k$ such that $M$ can be covered with $k$ open subsets each of which is homeomorphic to $Q \times [0,1)$. Recently, L. Montejano proved the following theorem concerning the relationship between $\text{cat}(P)$ and $C_z(P \times Q)$ (where $P \times Q$ is a $Q$-manifold [WE]):

**Theorem [MO1].** For every compact connected polyhedron $P$,

$$C_z(P \times Q) = \text{cat}(P) + 1.$$  

Obviously, $C_z(Q) = 2$. On the other hand, suppose that $M$ is a compact connected $Q$-manifold with $C_z(M) = 2$. By the triangulation theorem of [CH1], $M$ is homeomorphic to $P \times Q$ for some compact connected polyhedron $P$. Hence $\text{cat}(P) = 1$ by the above theorem, i.e., $P$ is contractible. Thus $M$ is homeomorphic to $Q$, since the Hilbert cube is the only contractible compact $Q$-manifold [CH1].

Extending to the noncompact case, we observe that if $P = \{\text{point}\}$,

$$C_z(P \times Q \times [0,1)) = 1 = \text{cat}(P).$$

Hence it is natural to ask if $C_z(P \times Q \times [0,1)) = \text{cat}(P)$ in general. In this note we answer the question affirmatively.

**Theorem 1.** For every compact polyhedron $P$,

$$C_z(P \times Q \times [0,1)) = \text{cat}(P).$$

For an arbitrary space $X$, let $\text{cat'}(X)$ denote the smallest integer $k$ such that $X$ can be covered by $K$ open sets each of which is null-homotopic in $X$. For a

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compact polyhedron $P$, it is easily seen that $\text{cat}(P) = \text{cat}'(P) = \text{cat}'(P \times Q)$. This observation, together with the triangulation theorem referred to above, shows that Montejano's result and our result may be formulated as follows:

**Corollary.** For every compact connected $Q$-manifold $M$, $C_z(M) = \text{cat}'(M) + 1$ and $C_z(M \times [0,1)) = \text{cat}'(M)$.

**Proof of Theorem 1.** It is easy to verify that $\text{cat}(P) \leq C_z(P \times Q \times [0,1))$. On the other hand, suppose that $\text{cat}(P) = k$. Let $P_1, P_2, \ldots, P_k$ be a null-homotopic cover of $P$. Denote $M = P \times Q \times \{0\} \subset P \times Q \times [0,1) = \tilde{M}$. Let $B_i = P_i \times Q \times \{0\}$. $B_i$ is a null-homotopic $Z$-set of $\tilde{M}$. By standard I-D topology we may assume that the cone of $B_i$ is imbedded in $\tilde{M}$ as a $Z$-set in such a way that the base coincides with $B_i$. Since the cone contracts to its vertex $v_i$ and $v_i$ has an open neighborhood $U_i$ homeomorphic to $Q \times [0,1)$, by $Z$-set unknotting there is a homeomorphism on $\tilde{M}$ taking $B_i$ into $U_i$. So, without loss of generality we may assume that $B_i \subset U_i$. Let $\{Q_n\}_{n=1}^{\infty}$ denote the collection of all end-faces of $Q$; that is, $Q_n = \pi_i^{-1}(0)$ or $\pi_i^{-1}(1)$ for some $i$, where $\pi_i$ is the $i$th-projection of $Q$ into the $i$th-factor. Let $A_{nm} = P_i \times Q_n \times [0, m/(m+1)]$, where $n, m = 1, 2, \ldots$, and let $A_i = P_i \times Q \times [0,1)$.

Each $A_{nm}$ is a $Z$-set in $\tilde{M}$ and is contractible to $P_i \times Q_n \times \{0\} \subset U_i$. We want to employ the engulfing apparatus of [CH1] to construct an open imbedding $h_i$ of $Q \times [0,1)$ into $\tilde{M}$ such that $\bigcup_{n,m} A_{nm} \subset h_i(Q \times [0,1))$. Let $g_i : Q \times [0,1) \to U_i$ be a homeomorphism. The key is to verify the following:

**Lemma A.** For any $Z$-set $K \subset P_i \times Q \times [0, r)$, $r < 1$, there is an open imbedding $g_i' : Q \times [0,1) \to \tilde{M}$ such that

$$g_i'(Q \times [0, \frac{1}{2}]) = g_i(Q \times [0, \frac{1}{2}]) \quad \text{and} \quad g_i'(Q \times [0, \frac{r}{2}]) \supset K.$$

**Proof of Lemma A.** Let $K' = K - g_i(Q \times [0, \frac{1}{2}])$, and assume $K' \neq \emptyset$. Since $P_i \times Q \times \{0\} \subset U_i$, $K'$ is homotopic to a subset of $g_i(Q \times [0, \frac{1}{2}])$ and the homotopy lies in the $Q$-manifold $N = \tilde{M} - g_i(Q \times [0, \frac{1}{2}])$. Since $g_i(Q \times [\frac{1}{2}, \frac{3}{4}])$ is a Hilbert cube and a $Z$-set in the Hilbert cube $g_i(Q \times [\frac{1}{2}, \frac{3}{4}]) \subset N$, using $Z$-set unknotting we may replace the homotopy by another homotopy $\{\phi_t\}$ of $K'$ into $N$ so that $\phi_0$ is identity, $\phi_1(K' \cap g_i(Q \times [\frac{1}{2}, \frac{3}{4}]))$ is identity for all $t$, $\phi_1(K') \subset g_i(Q \times [\frac{3}{4}, \frac{5}{4}])$ and for any $x \in K' - g_i(Q \times [\frac{3}{4}, \frac{5}{4}])$, $\{\phi_t(x)\}_t \subset N - g_i(Q \times [\frac{3}{4}, \frac{5}{4}])$. By $Z$-set unknotting there is a homeomorphism $f : N \to N$ such that $f|g_i(Q \times [\frac{1}{2}, \frac{3}{4}]) = \text{identity}$ and $f(K') \subset g_i(Q \times [\frac{1}{2}, \frac{3}{4}])$. Define $g_i'$ by $g_i'(Q \times [0, \frac{1}{2}]) = g_i(Q \times [0, \frac{1}{2}])$, $g_i'(Q \times [\frac{1}{2}, 1]) = f^{-1}g_i$. $g_i'$ is what we wanted.

By Lemma A and the engulfing lemma of [CH1] we can construct an open imbedding $h_i : Q \times [0,1)) \to \tilde{M}$ such that $\bigcup_{n,m} A_{nm} \subset h_i(Q \times [0,1))$. Let $V_i = h_i(Q \times [0,1))$ and $V = \bigcup_{i=1}^k V_i$. Then $V$ is an open set in $\tilde{M}$ containing the cap-set $P \times \bigcup_{n=1}^\infty Q_n \times [0,1)$; thus the complement $\tilde{M} - V$ is a $Z$-set [CH2]. It follows that $V$ and $\tilde{M}$ are homotopically equivalent, and therefore the manifold $V \times [0,1)$ and $\tilde{M}$ are homeomorphic. Since each $V_i \times [0,1)$ is homeomorphic to $Q \times [0,1)$, $C_z(\tilde{M}) = C_z(V \times [0,1)) \leq k$.

The theory extends to the following noncompact case.
THEOREM 2. For every connected \( Q \)-manifold \( M \),
\[
C_2(M \times [0, 1)) = \text{cat}'(M).
\]

PROOF. Since \( \text{cat}'(M) \) is a homotopy invariant, \( \text{cat}'(M) = \text{cat}'(M \times [0, 1)) \leq C_2(M \times [0, 1)) \). On the other hand, suppose \( \text{cat}'(M) = k \). It can be easily shown that \( M \) has a null-homotopic closed cover \( C_1, C_2, \ldots, C_k \). Consider \( M = M \times \{0\} \subset M \times [0, 1) \). As in the proof of Theorem A, each \( C_i \subset U_i \) where \( U_i \) is an open set homeomorphic to \( Q \times [0, 1) \). Let \( g_i : Q \times [0, 1) \to U_i \) be a homeomorphism. Denote \( A_{nm} = C_i \times Q_n \times [0, m/(m+1)] \). By the same argument as in the proof of Lemma A, there is an open imbedding \( g_i' : Q \times [0, 1) \to M \) such that
\[
g_i'(Q \times [0, 1]) = g_i(Q \times [0, 1]) \quad \text{and} \quad g_i'(Q \times [0, \frac{1}{2}]) \supset A_{nm}.
\]

Following the same argument as that of Theorem 1, we conclude that \( M \times [0, 1) \) is covered by \( k \) open sets each of which is homeomorphic to \( Q \times [0, 1) \). Hence \( C_2(M \times [0, 1)) \leq k \).

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