ABSTRACT. We construct within ZFC a family of pairwise nonhomeomorphic
dense-in-itself rigid extremally disconnected compact separable spaces. Any
such space is determined by a sequence of weak $P$-points of $\mathbb{N}^*$ having different
types.

Introduction. In their review paper, van Douwen, Monk and Rubin [DMR]
state that examples of rigid Boolean algebras published in the literature are rather
special and are not easily described. They also ask for some "natural" examples.
In this paper the authors have constructed within ZFC a large family of separable,
compact Hausdorff, extremally disconnected perfect (no isolated points) rigid topo-
logical spaces. Thanks to the duality theory of Stone, this translates in Boolean
algebra to a family of nonatomic complete rigid Boolean algebras. Moreover, their
structure is so simple and natural, as to be deemed canonical examples. We hope
that, with these examples, we have answered the question of [DMR].

Our starting point is a known technique of construction of a family of Tychonov
rigid spaces. (See for example Gubbi [G], or Kannan and Rajagopalan [KR].) Al-
though by a Stone space is usually meant a compact Hausdorff totally disconnected
space, for want of a suitable short word, we mean in the sequel an extremally dis-
connected (E.D.) compact Hausdorff space. A topological space is rigid if its only
autohomeomorphism is the identity. All other terms are standard and can be easily
located in standard texts.

Let Seq be the set of all finite sequences of natural numbers, $\mathbb{N}$. For every $s \in \text{Seq}$
let $\xi_s$ be a filter on $\mathbb{N}$ containing the Fréchet filter on $\mathbb{N}$. Define a topology $\mathcal{T}$ on
Seq by the rule: $V \subset \text{Seq}$ is open iff for every $s \in V$, the set \( \{n: s \uparrow n \in V\} \in \xi_s \).
Here $s \uparrow n$ denotes the concatenation of $s$ by $n$ [J]. It is easy to verify that Seq with
this topology is Tychonoff, 0-dimensional and perfect. If $\xi_s$ is the Fréchet filter
for every $s \in \text{Seq}$, then Seq with this topology is the space $S_\omega$ of [AF]. If $\xi_s$ is an
ultrafilter for every $s \in \text{Seq}$ then Seq with $\mathcal{T}$ is an E.D. space.

In order to facilitate further discussion, let us consider some special subspaces
of Seq. Let $L_n$ denote the set of all sequences of length $n$ (this is also the $n$th
level of the tree of all sequences) and let $T_n$ stand for the subspace of all sequences
of length $\leq n$, $n \in \omega$. The sequence of length 0 is denoted $s_0$, the base point for
the space constructed above. Observe that $T_n$ is a closed nowhere dense subspace

Received by the editors June 13, 1986 and, in revised form, December 1, 1986. This paper was
1980 Mathematics Subject Classification (1985 Revision). Primary 54G05; Secondary 06E15.
Key words and phrases. Rigid space, weak $P$-point, $\mathbb{N}^*$.

The third author expresses his gratitude to the Department of Mathematical and Computer
Sciences of the Youngstown State University for its hospitality during the preparation of this work.
of $\text{Seq}$ and $L_n$ is discrete and dense in $T_n$. We shall denote by $\mathcal{G}_\omega$ the space $(\text{Seq}, T)$ determined by choosing $\xi$'s to be free ultrafilters on $\omega$. The following fact concerning E. D. Hausdorff spaces is quite obvious: if $X$ is an E.D. Hausdorff space, $A$ and $B$ are $\sigma$-compact subsets of $X$ such that $\text{cl} A \cap B = \emptyset = A \cap \text{cl} B$, i.e., $A$ and $B$ are separated, then $\text{cl} A \cap \text{cl} B = \emptyset$. We shall now prove some facts concerning the separability properties of $\mathcal{G}_\omega = \beta \mathcal{G}_\omega - \mathcal{G}_\omega$, $\beta \mathcal{G}_\omega$ standing for the Stone-Cech compactification of $\mathcal{G}_\omega$.

**Lemma 1.** Let $U_1, U_2, \ldots$ be a sequence of clopen subsets of $\beta \mathcal{G}_\omega$ such that (1) $s_0 \notin U_1$, and (2) $U_n \cap ((L_1 - U_1) \cup (L_2 - U_2) \cup \cdots \cup (L_{n-1} - U_{n-1})) = \emptyset$ for $n > 1$. Then $s_0 \notin \text{cl}(U_1 \cup U_2 \cup \cdots)$.

**Proof.** It is enough to prove that $V = \mathcal{G}_\omega - (U_1 \cup U_2 \cup \cdots)$ is a neighborhood of $s_0$ in $\mathcal{G}_\omega$. First, note that $s_0 \in V$. Let now $t \in V$ and let us suppose that the length of $t$ is $m \geq 0$.

Since $t \in V$, $t \notin U_1 \cup \cdots \cup U_{m+1}$. Therefore $\mathcal{G}_\omega - (U_1 \cup \cdots \cup U_{m+1})$ is an open neighborhood of $t$. Hence $A = \{n \in \mathbb{N} : t^n \in \mathcal{G}_\omega - (U_1 \cup \cdots \cup U_{m+1})\} \in \xi_t$. We shall show that for any $n \in A$, $t^n \in V$. If not, then $t^n \in U_k$ for some $k$. By assumption, $U_k \cap ((L_1 - U_1) \cup \cdots \cup (L_{k-1} - U_{k-1})) = \emptyset$. In particular, $U_k \cap (L_{m+1} - U_{m+1}) = \emptyset$. But if $t \in L_n$, then $t^n \in L_{m+1}$, so $t^n \in L_{m+1} - U_{m+1}$ for $n \in A$. Hence $U_k \cap (L_{m+1} - U_{m+1}) \neq \emptyset$; a contradiction.

**Lemma 2.** Let $K$ be a $\sigma$-compact subset of $\beta \mathcal{G}_\omega$ such that $K \cap \text{cl} L_n = \emptyset$ for every $n$. Then $\text{cl} K$ is contained in $\mathcal{G}^*_\omega$.

**Proof.** Let $K = \bigcup K_n$, $K_n$ being compact. Let $U_n$ be a clopen neighborhood of $K_1 \cup \cdots \cup K_n$ disjoint with $\text{cl} L_1 \cup \cdots \cup \text{cl} L_n$. We claim that $U_1, U_2, \ldots$ satisfy Lemma 1.

Indeed $U_1 \cap \text{cl} L_1 = \emptyset$ and $s_0 \notin \text{cl} L_1$. Now if $n > 1$, then

$$U_n \cap ((L_1 - U_1) \cup \cdots \cup (L_{n-1} - U_{n-1})) \subset U_n \cap (\text{cl} L_1 \cup \cdots \cup \text{cl} L_{n-1}) = \emptyset.$$  

By Lemma 1 $s_0 \notin \text{cl}(U_1 \cup U_2 \cup \cdots)$. Hence $s_0 \notin \text{cl} K$. A similar proof works for every $s \in \mathcal{G}_\omega$, and this completes the proof. □

A point of a space is said to be a weak $P$-point if it is not an accumulation point of any countable subset of the space.

**Theorem 1.** Suppose that for every $t \in \mathcal{G}_\omega$, $\xi$ is chosen to be a weak $P$-point of $\mathbb{N}^*$. Then the closure of every countable subset of $\mathcal{G}^*_\omega$ is contained in $\mathcal{G}^*_\omega$.

**Proof.** Let $D$ be a countable subset of $\mathcal{G}^*_\omega$. Put $D_1 = \text{cl} L_1 \cap D$ and $D_{n+1} = \text{cl} L_{n+1} \cap (D - (D_1 \cup \cdots \cup D_n))$ for every $n$. Note that $D_{n+1}$ is disjoint with $\text{cl} L_n$. Because $\xi_{s_0}$ is a weak $P$-point of $\mathbb{N}^*$, $s_0 \notin \text{cl} D_1$. Let $U_1$ be a clopen set in $B \mathcal{G}_\omega$, such that $U_1 \supset D_1$ and $s_0 \notin U_1$. Suppose we have defined $U_1, U_2, \ldots, U_n$ satisfying:

1. $D_k \subset U_k$,
2. $U_k \cap [\text{cl}(L_1 - U_1) \cup \cdots \cup \text{cl}(L_{k-1} - U_{k-1})] = \emptyset$

for $k = 2, \ldots, n$.

Now observe that $\text{cl} D_{n+1} \cap [\text{cl}(L_1 - U_1) \cup \cdots \cup \text{cl}(L_n - U_n)] = \emptyset$. In order to see this, note first that since $D_{n+1} \subset \text{cl} L_{n+1}$, $\text{cl} D_{n+1} \cap L_{n+2} = \emptyset$. For every $s \in L_{n+1}$ the set $\{s^n : n \in \mathbb{N}\}$ is a discrete subspace of $\mathcal{G}_\omega$. Since $\xi_s$ is a weak $P$-point of $\beta \mathbb{N}$ and the induced topology on $\text{cl}\{s^n : n \in \mathbb{N}\}$ coincides with that for $\beta \mathbb{N}$ (by
identifying $n$ with $s^n$), there is an $A \in \beta_\omega$ such that $\text{cl}\{s^n: n \in A\} \cap D_{n+1} = \emptyset$. Hence $\text{cl}\{s^n: n \in A\} \cap \text{cl}_\omega D_{n+1} = \emptyset$, $D_{n+1}$ and $\{s^n: n \in A\}$ being two countable separated subsets of $\beta_\omega$. In consequence $s \notin \text{cl}_\omega D_{n+1}$ for every $s \in L_{n+1}$, i.e., $\text{cl}_\omega D_{n+1} \cap L_{n+1} = \emptyset$. Arguing as above for every $t \in L_n$, we get that $\text{cl}_\omega D_{n+1} \cap L_n = \emptyset$. At the beginning of the proof we have noted that $D_{n+1} \cap \text{cl} L_n = \emptyset$. Hence $\text{cl}_\omega D_{n+1} \cap \text{cl} L_n = \emptyset$, $D_{n+1}$ and $L_n$ being two separated subsets of $\beta_\omega$. Now we can find a clopen set $U_{n+1} \supset D_{n+1}$ such that $U_{n+1} \cap \text{cl}(L_1 - U_1) \cup \cdots \cup \text{cl}(L_n - U_n) = \emptyset$. In fact, any clopen set containing $D_{n+1}$ and disjoint with $L_n$ will possess this property. Using Lemma 1, $s_0 \notin \text{cl}(U_1 \cup U_2 \cup \cdots)$. But $U_1 \cup U_2 \cup \cdots$ contains $D_1 \cup D_2 \cup \cdots = D \cap \bigcup\{\text{cl} L_n: n \in \mathbb{N}\}$. In virtue of Lemma 2, $s_0 \notin \text{cl}(D \cup \bigcup\{\text{cl} L_n: n \in \mathbb{N}\})$. Therefore $s_0 \notin \text{cl} D$. This proof can be repeated for every $t \in \beta_\omega$, so the proof of the theorem is complete. □

REMARK 1. If we choose $P$-points instead of weak $P$-points, then one can extend Theorem 1 to $\sigma$-compact subsets of $\beta_\omega$.

REMARK 2. Theorem 1 says that for a suitable choice of ultrafilters $\xi_s$, the space $\beta_\omega$ is nowhere separable. The question about nonseparability of $\beta_\omega$ was communicated to the second author by M. Rajagopalan.

Two ultrafilters $\xi_1, \xi_2 \in \beta \mathbb{N}$ are of the same type if there is a permutation $f$ of $\mathbb{N}$ such that the extension of $f$ to $\beta \mathbb{N}$ carries $\xi_1$ onto $\xi_2$.

THEOREM 2. Suppose that $\xi_t$ is chosen to be a weak $P$-point of $\mathbb{N}^*$ for every $t \in \mathcal{G}_\omega$ and $\xi_t$ and $\xi_s$ are of different types for $t \neq s, t, s \in \mathcal{G}_\omega$. Then the space $\beta \mathcal{G}_\omega$ is rigid.

PROOF. Let $h: \beta \mathcal{G}_\omega \to \beta \mathcal{G}_\omega$ be an autohomeomorphism and put $D = h(\mathcal{G}_\omega) \cap \beta \mathcal{G}_\omega$. Then $D$ is a countable subset of $\beta \mathcal{G}_\omega$. In virtue of Theorem 1, $\text{cl} D \subset \beta \mathcal{G}_\omega$ and therefore $\text{cl} D$ is a nowhere dense closed subset of $\beta \mathcal{G}_\omega$. Hence $\mathcal{G}_\omega - h^{-1}(\text{cl} D)$ is an open dense subspace of the space $\mathcal{G}_\omega$ and $\mathcal{G}_\omega - h^{-1}(\text{cl} D) \subset \mathcal{G}_\omega$. We shall show that $h(t) = t$ for every $t \in \mathcal{G}_\omega - h^{-1}(\text{cl} D)$.

Let $t \in \mathcal{G}_\omega - h^{-1}(\text{cl} D)$ and let $s = h(t)$. The set $A = \{n \in \mathbb{N}: t^n \in \mathcal{G}_\omega - h^{-1}(\text{cl} D)\}$ is in $\xi_t$ and the set $B = \{t^n: n \in A\}$ is a discrete subset of $\mathcal{G}_\omega - h^{-1}(\text{cl} D)$ containing $t$ and no other points in its closure. Hence $h(B)$ is a discrete subset of $\mathcal{G}_\omega$ containing $s$ and no other points in its closure. This implies that $\{n \in \mathbb{N}: s^n \in h(B)\} \in \xi_s$. Now consider a map $f: A \to \mathbb{N}$ defined as follows: if $n \in A$, then $f(n) = m$, where $m$ is such that $h(t^n) = s^m$. The map $f$ is 1-1. If $\bar{f}$ is the unique extension of $f$ to $\beta A$, then $\bar{f}(\xi_t) = \xi_s$, because $h(t) = s$. This shows that the ultrafilters $\xi_t$ and $\xi_s$ are of the same type and in consequence that $t = s = h(t)$. Since the restriction of $h$ to some dense subset of $\beta \mathcal{G}_\omega$ is the identity map, $h$ has to be the identity map as well. □

Within ZFC, K. Kunen [K] has shown that there are $2^c$ different types among weak $P$-points of $\mathbb{N}^*$. Each selection of countably many such types produces a rigid separable Stone space. Different selections give nonhomeomorphic spaces. So finally we have

MAIN THEOREM. There exist $2^c$ pairwise nonhomeomorphic rigid separable Stone spaces.

ADDENDUM. We were informed by Jan van Mill that E. van Douwen has obtained independently similar results and P. Simon has also known some results concerning the spaces $(\mathcal{S}(\mathbb{N}), T)$. 

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