ON THE JACOBSON RADICAL OF SOME ENDOMORPHISM RINGS
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ABSTRACT. In this note we deal with a question raised by R. S. Pierce in 1963: Determine the elements of the Jacobson radical of the endomorphism ring of a primary abelian group by their action on the group. We concentrate on separable abelian \( p \)-groups and give a counterexample to a conjecture of A. D. Sands. We also show that the radical can be pinned down if the endomorphism ring is a split-extension of its ideal of all small maps.

Introduction. All groups in this note are abelian \( p \)-groups for some fixed but arbitrary prime \( p \). Our notations are standard as in [F]. It is known that the endomorphism ring \( \text{End}(A) \) of an abelian \( p \)-group \( A \) determines the group up to isomorphism. R. Pierce [P] raised the question of describing the Jacobson radical \( J(\text{End}(A)) \) of \( \text{End}(A) \) by its action on the group. This problem was solved by W. Liebert [L], J. Hausen [H] and Hausen-Johnson [HJ] for \( \Sigma \)-cyclic, torsion-complete and sufficiently projective \( p \)-groups. (For a separable \( p \)-group sufficiently projective is the same as \( \omega_1 \)-separable.) If \( A \) is a (separable) \( p \)-group, let \( H(A) = \{ \varphi \in \text{End}(A) | |x| < |x\varphi| \text{ for all } 0 \neq x \in A[p] \} \) be the ideal of all maps acting height increasing on the socle of \( A \), and let \( C(A) \) be the ideal of all elements of \( \text{End}(A) \) mapping each Cauchy sequence in \( A[p] \) onto a convergent one. (For \( x \in A \), \(|x|\) denotes the \( p \)-height of \( x \) in \( A \) and topological notations refer to the \( p \)-adic topology.) If \( A \) is torsion-complete, \( J(\text{End}(A)) = H(A) \), if \( A \) is \( \Sigma \)-cyclic or \( \omega_1 \)-separable, \( J(\text{End}(A)) = H(A) \cap C(A) \), and \( H(A) \cap C(A) \subset J(\text{End}(A)) \) for all separable \( p \)-groups (cf. [S]). The purpose of this paper is to show that \( J(\text{End}(A)) \) is in general not equal to \( C(A) \cap H(A) \) for separable \( p \)-groups \( A \). We will use that \( J(\text{End}(A)) \cap E_s(A) = E_s(A) \cap H(A) \), where \( E_s(A) \) is the ideal of all small endomorphisms of \( A \) (cf. [S]). Recently, many complicated \( p \)-groups have been constructed in [DG1, DG2, CG]. All these groups enjoy the property that \( \text{End}(A) \) is a split extension of \( E_s(A) \), i.e. \( \text{End}(A) = R \oplus E_s(A) \) for some subring \( R \) of \( \text{End}(A) \). The way these groups are constructed, \( R \cap H(A) = pR \) and \( \overline{H}_R(A) = \overline{H}_R(A) \), i.e. if \( r \in R - H(A) \), then for all \( n \) there is \( 0 \neq x \in p^n A[p] \) such that \( x \) and \( xr \) have the same height. In this situation Theorem 1 below implies

\[
J(\text{End}(A)) = (J(R) \cap H(A)) \oplus (E_s(A) \cap H(A))
\]

and we have \( J(\text{End}(A)) = H(A) \cap C(A) \) for these groups. We will construct a ring \( R \) and use the realization result in [C] to obtain a separable \( p \)-group \( A \) such that

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End\((A) = R \oplus E_s(A)\) and \((J(R) - pR) \cap H(A)\) and \((R - J(R)) \cap H(A)\) are both not empty. Since \(C(A) \cap R = pR\) for this group \(A\), we have that \(H(A) \cap C(A)\) is a proper subset of \(J(\text{End}(A))\). This makes it hard to believe that Pierce’s question has a positive answer for all separable p-groups.

**The construction.** Let \(A\) be a separable p-group and \(E = \text{End}(A)\) the endomorphism ring of \(A\). If \(S \subseteq E\), and \(U \subseteq A[p]\) are subsets, define \(H_S(A, U) = \{ \varphi \in S \mid \exists x \in U, \ |x| = |x\varphi| \}\). Observe that \(H_E(A, U) \cap S = H_S(A, U)\) and if \(U \subseteq V\) we have \(H_S(A, U) \subseteq H_S(A, V)\). We also define \(H(A) = E - H_E(A, A[p])\), \(H_S(A) = \bigcap_n H_S(A, p^nA[p])\). Moreover, let \(H_S(A) = S \cap H(A)\) and \(H_S(A, U) = S - H_S(A, U)\).

**Theorem 1.** Let \(A\) be a separable p-group such that \(E = \text{End}(A) = R \oplus E_s(A)\) is a split extension of \(E_s(A)\) and \(H_R(A) = H_R(A)\). Then \(J(E) = H_J(R)(A) \oplus H_E(A)(A)\).

**Proof.** Let \(r \in R\), and \(\sigma \in E_s(A)\) with \(r + \sigma \in J(E)\). Then \((r + \sigma)t\) is right quasi-regular for all \(t \in R\) and, since \(E = R \oplus E_s(A)\), the element \(rt\) is right quasi-regular in \(R\) as well as \(r \in J(R)\). Now suppose \(r \notin H_J(R)(A)\). Since \(\sigma\) is small and \(H_R(A) = H_R^+(A)\), we find \(n < \omega\) and \(0 \neq x \in p^nA[p]\) such that \(x\sigma = 0\) and \(|x| = |x\sigma|\). Let \(F\) be a finite summand of \(A\) containing \(x\sigma\) and let \(\rho: A \to F\) be the natural projection. Then \(x = x\rho\) and \(|x| = |x\rho| = |x\sigma| = |x(r + \sigma)p|\). Since \(A = F\) is finite, \(\rho \in E_s(A)\) and hence \((r + \sigma)p \in J(E) \cap E_s(A) = E_s(A) \cap H(A)\) (cf. [S]). This contradicts the above equation of heights and we conclude \(r \in H_J(R)(A)\). Now let \(t \in R\), \(\sigma \in E_s(A)\) and \(r \in H_J(R)(A)\). Then there is \(s \in R\) such that \((1 - rt)s = 1\). This implies \((1 - r(t + \sigma))s = (1 - rt)s - r\sigma s = 1 - r\sigma s\). Since \(\sigma \in E_s(A)\) and \(r \in H_J(R)(A)\) we have that \(r\sigma s \in H_E(A)(A) \subseteq J(E)\) and there is \(\tau \in E\) with \((1 - r\sigma s)\tau = 1\). This implies \((1 - r(t + \sigma))\sigma \tau = 1\) and \(r \in J(E)\). We obtain \(H_J(R)(A) \subseteq J(E) \subseteq H_J(R) \oplus E_s(A)\) which together with \(J(E) \cap E_s(A) = H_E(A)(A)\) implies the desired equation.

We now construct our ring:

Let \(\omega\) be the set of natural numbers including 0 and let

\[
B = \bigoplus_{i \in \omega} (f_i) \oplus \bigoplus_{i \in \omega} (g_i) \oplus \bigoplus_{i \in \omega} (h_i)
\]

be a \(\Sigma\)-cyclic p-group with \(\exp(f_i) = i + 1 = \exp(g_i)\) and \(\exp(h_i) = i + 2\). We define elements \(\alpha, \beta, \gamma \in \text{End}(B)\) by setting \(f_i\alpha = p f_{i+1}\), \(f_i\beta = g_i\) and \(f_i\gamma = ph_i\), and \(\alpha, \beta, \gamma\) are 0 on the \(g_i\)'s and \(h_i\)'s. Let \(S = \langle 1, \alpha, \beta, \gamma \rangle\) be the subring of \(\text{End}(B)\) generated by these elements and \(R = \hat{S}\) be the p-adic completion of \(S\). We have the following relations:

1. \(\beta\alpha = \gamma\alpha = \beta^2 = \gamma^2 = \beta\gamma = \gamma\beta = 0\).
2. \(r = \sum_{i=0}^n \alpha^i a_i + \sum_{i=0}^m \alpha^i b_i + \sum_{i=0}^k \alpha^i c_i\) with \(a_i, b_i\) and \(c_i\) integers.

Therefore each element \(x \in R = \hat{S}\) has a unique representation:

\(x = \sum_{i=0}^\infty \alpha^i a_i + \sum_{i=0}^\infty \alpha^i b_i + \sum_{i=0}^\infty \alpha^i c_i\) where \(\{a_i\}, \{b_i\}\) and \(\{c_i\}\) are p-adic zero-sequences in \(F\), the ring of p-adic integers. Let \(I\) be the set of all \(x \in R\) with all \(a_i\)'s being 0. This is the ideal of \(R\) generated by \(\beta\) and \(\gamma\). An easy computation shows:
(4) Let \( x \in R \) be as in (3). Then \( \exp(f_k x) = \max_{i \in \omega} \{k + 1 - |a_i|, k + 1 - |b_i|, k + 1 - |c_i|\} \). Here the max is defined to be 0 if all numbers in the set are < 0. Because of (3), \( \{\alpha^i \mid i < \omega\} \) is linearly independent and we have

(5) \( R/I \cong (J_p[\alpha])^{-\mathbb{Z}} \), the \( p \)-adic completion of the polynomial ring \( J_p[\alpha] \).

This implies

(6) \( R/(pR + I) \cong GF(p)[\alpha] \), the polynomial ring over \( GF(p) \).

This implies \( pR \subset J(R) \subset pR + I \).

(7) Let \( j < k + 1 \). Then \( |p^j f_k x| > j = |p^j f_k| \) for all \( x \in I \), the ideal generated by \( \beta \) and \( \gamma \), i.e. \( I \subset H(\overline{B}) \).

This follows from a straightforward computation using that \( B \) is \( \Sigma \)-cyclic. Observe that \( I^2 = 0 \).

(8) \( J(R) = pR + I \).

We want \( R \) to be pure in \( E(\overline{B}) \), \( \overline{B} \) the torsion-completion of \( B \). We show a little more:

(9) \( R \oplus E_s(\overline{B}) \) is pure in \( E(\overline{B}) \). (One needs this to do a “Black Box” construction; cf. [CG].)

To prove (9), let \( \varphi \in E(\overline{B}), r \in R, \sigma \in E_s(\overline{B}) \) and \( n \in \omega \) with \( p^n \varphi = r + \sigma \). Since \( \sigma \) is small, there is \( k \in \omega \) such that \( p^{k-n-1}f_k p^n \varphi = p^{k-n-1}f_k r \) and \( p^2 p^{k-1}f_k \varphi = 0 \). Hence \( p^{k-n+1}f_k r = 0 \). Now let \( r \) be represented as in (3) and apply (4) to obtain:

\[
k - n + 1 \geq \exp(f_k r) = \max_{i \in \omega} \{k + 1 - |a_i|, k + 1 - |b_i|, k + 1 - |c_i|\}.
\]

This implies \( k - n + 1 \geq k + 1 - |a_i| \) and \( |a_i| \geq n \). The same holds for the \( b_i \)'s and \( c_i \)'s. Therefore \( r \in p^n R \) and \( r = p^n s \) for some \( s \in R \). Thus \( p^n(\varphi - s) = \sigma \) is small which implies \( \varphi - s \) is small and \( \varphi \in R \oplus E_s(\overline{A}) \). \( \square \)

(10) Let \( A \) be a pure subgroup of \( \overline{B} \) containing \( B \) and \( \varphi \in H_{E(A)}(A, B[p]) \). Then \( \varphi \in H_{E(A)}(A, A[p]) \).

Let \( a \in A[p] - B[p] \). Since \( A/B \) is divisible, there is \( b \in B, y \in A \) such that \( a = b + p^n y \) where \( n = |a| \). Then \( |a| = |b| \) and \( |a \varphi| = |(b + p^n y) \varphi| \geq n + 1 > n = |a| \) since \( \varphi \in H_{E(A)}(A, B[p]) \). This inequality shows \( \varphi \in H_{E(A)}(A, A[p]) \). \( \square \)

Now we apply A. L. S. Corner’s result [C, Theorem 2.1] and obtain a pure subgroup \( A \) of \( \overline{B} \) containing \( B \) and \( \text{End}(A) = R \oplus E_s(A) \). (Observe that (4) implies that condition (C) of [C, Theorem 2.1] holds.) The ring \( R \) is constructed to satisfy \( \text{H}_R(A) = pR + aR + R \gamma \). Moreover we have \( \alpha \in \text{h}_R(A) - J(R) \), \( \beta \in J(R) - \text{H}_R(A) \) and \( \gamma \in \text{H}_{J(R)}(A) - pR \). Observe that \( \gamma \notin C(A) \), since otherwise \( \overline{B}[p] \gamma \subset A \).

In order to see that this is absurd, we have to look into Corner’s proof [C] of his Theorem 2.1: Recall that for a positive integer \( e \) an element \( x \in \overline{B} \) is \( e \)-strong if \( xr = 0 \) implies \( r \in p^e R \) for \( x \in \overline{B}[p^e] \) and \( r \in R \). Our group \( A \) is one of Corner’s \( \text{G}_\rho \) (cf. [C, Theorem 2.1]). Corner observes [C, p. 285, line -6] that each of his \( \text{G}_\rho \) contains for any \( e \) an \( e \)-strong element \( x \) such that for \( \sigma \neq \rho \) we have \( \text{G}_\sigma \cap xR = 0 \). For \( e = 1 \), \( x \in \overline{B}[p] \) and since \( \gamma \notin pR \) we conclude \( x \gamma \neq 0 \). This shows that \( \overline{B}[p] \gamma \) is not contained in any \( \text{G}_\sigma \) (\( = A \)). Now Theorem 1 applies and we have that \( \text{J}(\text{End}(A)) = \text{H}_{J(R)}(A) \oplus H_{E_s(A)}(A) \) is not contained in \( C(A) \cap H(A) \) since \( \gamma \) is not and also \( \text{H}_R(A) \) is not contained in \( J(R) \). So if we want to describe the elements of \( \text{J}(\text{End}(A)) \) by their action on \( A \), we have to find the elements in \( J(R) \cap H(A) \), which means we must be able to recognize the elements of \( J(R) \). There is much freedom for the way an element of \( J(R) \) can operate on \( A \). We answer a question
in [S] by summarizing part of our discussion in

**Theorem 2.** There exists a separable p-group $A$ such that $J(\text{End}(A))$ is larger than $H(A) \cap C(A)$.

**Remark.** If we want to have larger groups $A$ realizing $R$, we may employ Shelah's "Black Box" and a construction very similar to the one in [CG]. We would like to mention again that all the p-groups constructed in [CG, DG1 or DG2] satisfy $H(A) \cap J(R) = pR$.

**References**


