CHARACTERIZATION OF SCHWARTZ SPACES
BY THEIR HOLOMORPHIC DUALS

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ABSTRACT. Let U be an open subset of a locally convex space E, and let $H_c(U, F)$ denote the vector space of holomorphic functions into a locally convex space F, endowed with continuous convergence. It is shown that if F is a semi-Montel space, then the bounded subsets of $H_c(U, F)$ are relatively compact. Further it is shown that E is a Schwartz space iff the continuous convergence structure on the algebra $H(U)$ of scalar-valued holomorphic functions on U, coincides with local uniform convergence. Using this, an example of a nuclear Fréchet space E is given, such that the locally convex topology associated with continuous convergence on $H(E)$ is strictly finer than the compact open topology. Thus, the behavior of the space $H_c(E)$ differs in this respect from that of its subspace L_cE of linear forms and that of its superspace $C_c(E)$ of continuous functions.

Introduction. In [11] H. Jarchow has proved that a locally convex space (lcs) E is Schwartz if and only if continuous convergence and local uniform convergence coincide on the dual of E. It is natural to ask if there is a holomorphic analogue of this result: Is a space E Schwartz if and only if continuous convergence and local uniform convergence coincide on a space $H(U)$ of holomorphic functions on some open subset U of E? We give a positive answer to this question using a result, closely related to Jarchow's, for continuous m-homogeneous polynomials [5]. Further we prove that the space $H(U)$ has the Montel property (bounded sets are relatively compact) when endowed with continuous convergence.

For spaces of linear forms (on locally convex spaces) as well as for spaces of continuous functions (on e.g. completely regular spaces) it is known that the locally convex topology associated with continuous convergence is the compact-open topology (cf. [1 and 10]). Using the above holomorphic characterization of Schwartz spaces, we provide an example which shows that spaces of holomorphic functions behave quite differently: There exists a nuclear Fréchet space E, such that the locally convex topology associated with continuous convergence on $H(E)$ is strictly finer than the compact-open topology.

We recall some notation and definitions. All vector spaces in this paper are complex. A function $f: U \to F$ into a convergence vector space (cvs) F is *Gateaux-holomorphic* if the function $\lambda \mapsto l \circ f(x + \lambda h)$ is holomorphic in a neighborhood of zero for each $x \in U$, $h \in E$, and $l \in LF$. It is *holomorphic* if it is Gateaux-holomorphic and continuous. Let $H(U, F)$ be the vector space of holomorphic

functions $f: U \to F$. When endowed with continuous convergence [1] or with the associated equable convergence [8] (which coincides with local uniform convergence if F is normed) it will be denoted by $H_c(U, F)$ and $H_e(U, F)$, respectively. A net $(f_k)_k$ converges to zero in $H_c(U, F)$ if and only if for each $x \in U$ and each net $(x_l)_l$, which converges to x, the net $(f_k(x_l))_{kl}$ converges to zero in F. The net $(f_k)_k$ converges to zero in $H_e(U, F)$ if and only if for each $x \in U$ and each $(x_l)_l$, which converges to x, there is a filter $\mathcal{G}$ converging to zero in F, such that for each $G \in \mathcal{G}$ exists an l_0, such that $q_G(f_k(x_l))$ converges to 0 uniformly on $\{x_l : l \geq l_0\}$. For cvs E and F, let $P_c(^mE, F)$ and $P_e(^mE, F)$ be the subspaces of $H_c(U, F)$ and $H_e(U, F)$, respectively, consisting of all continuous m-homogeneous polynomials. A cvs E is L_c-*embedded* (or L_e-*embedded*) if the mapping $j_E: E \to L_cL_cE$ (or $j_E: E \to L_eL_eE$), $j_E(x)l = l(x)$ into the second dual is an embedding [2]. All Hausdorff lcs are L_α-embedded ($\alpha = c, e$). Further $H_c(U, F)$ and $P_c(^mE, F)$, $m \in \mathbf{N}$, are L_c-embedded spaces and $H_e(U, F)$ and $P_e(^mE, F)$, $m \in \mathbf{N}$, are L_e-embedded spaces. A subset B (or a filter $\mathcal{F}$) in a cvs E is said to be *bounded* (or *quasi-bounded* [8]), if $\mathbf{V}B \to 0$ (or $\mathbf{V}\mathcal{F} \to 0$) in E, where $\mathbf{V}$ denotes the filter of neighborhoods of 0 in $\mathbf{C}$. Note that a subset (or a filter) in $H(U, F)$ is bounded (or quasi-bounded) in $H_c(U, F)$ if and only if it is bounded (or quasi-bounded) in $H_e(U, F)$. Clearly all convergent filters in a cvs are quasi-bounded. The adherence $a(B)$ of a subset B of E is by definition the set $\{x \in E : \exists$ a filter $\mathcal{F} \to x$ with $B \in \mathcal{F}\}$. A subset B of E is said to be *closed* if $a(B) = B$, *compact* if every ultrafilter $\mathcal{U}$ with $B \in \mathcal{U}$ converges in B, and *relatively compact* if $a(B)$ is compact.

The Montel property. First we show that $H_c(U, F)$ has the Montel property, when F is a semi-Montel space. The proof is based on Theorem 1, the proof of which can be found in [6] (cf. also [9]). If X and Y are convergence spaces, $C_c(X, Y)$ will denote the set of continuous functions from X into Y, endowed with continuous convergence.

1. THEOREM. *Let X be a convergence space and Y a uniform space, and let $B \subseteq C(X, Y)$. Then B is relatively compact in $C_c(X, Y)$ if and only if it is equicontinuous and, for each $x \in X$, $B(x)$ is relatively compact in Y.*

2. PROPOSITION. *Let U be an open subset of an lcs E and let F be a semi-Montel space. Then every bounded subset B of $H(U, F)$ is relatively compact in $H_c(U, F)$.*

PROOF. Let B be bounded in $H_c(U, F)$. Then, for each $x \in U$ and each circled convex 0-neighborhood V in F, there exists a 0-neighborhood W in E with $x + W \subseteq U$ and $\varepsilon > 0$, such that $\varepsilon B(x + W) \subseteq V$. Thus B is pointwise bounded. Further, by Cauchy's inequality,

$$\sup_{h \in W} q_V(1/m! \, \hat{d}^m f(x)h) \leq q_V(f(x + W)) \leq \varepsilon$$

for $f \in B$ (q_V denotes the gauge of V). Now

$$q_V(f(y) - f(x)) \leq \sum_{m=1}^{\infty} q_V\left(\frac{1}{m!} \hat{d}^m f(x)(y - x)\right) \leq \varepsilon \sum_{m=1}^{\infty} \delta^m = \varepsilon\delta(1 - \delta)^{-1}$$

for arbitrary $y \in x + \delta W$, $\delta < 1$. Hence $f(y) - f(x) \in \varepsilon\delta(1 - \delta)^{-1}V$ for each $f \in B$, i.e. B is equicontinuous. Since F is semi-Montel, the set $B(x)$ ($x \in U$), is

relatively compact in F. Theorem 1 yields the proposition, since $H(U, F)$ is closed in $C_c(U, F)$ according to Theorem 2.1 in [**4**].

REMARKS. (a) In L_c-embedded spaces, and thus also in L_e-embedded spaces, relatively compact subsets are bounded (cf. [**3**]). In the proof for Proposition 2 we showed that bounded subsets of $H_c(U, F)$ are equicontinuous and pointwise bounded. The converse statement is also true, as is easily seen.

(b) Let U^{lc} denote the locally compact convergence space associated with U. Now $H_{HY}(U, F) \cong H_c(U^{\mathrm{lc}}, F)$, where the space $H_{HY}(U, F)$ of hypoanalytic functions is endowed with the compact-open topology (cf. [**4**]). Since, under the assumptions in (d) below, a subset of $H(U)$ is bounded in $H_c(U)$ if and only if it is bounded in $H_{co}(U)$ (Proposition 3.5 in [**4**]), we obtain the well-known results [**7**]:

(c) If U is an open subset of an lcs E, and F is a semi-Montel space, then $H_{HY}(U, F)$ is a semi-Montel space.

(d) Let E be a metrizable lcs or the strong dual of a Fréchet-Montel space and U an open subset of E. Then $H_{co}(U)$ is semi-Montel.

Characterization of Schwartz spaces. We generalize a result in [**1**] to the case that we have a cvs F as target space:

3. LEMMA. *For a compact topological space X and an L_c-embedded cvs F, a net $(f_i)_i$ converges to zero in $C_c(X, F)$ if and only if it converges to zero uniformly on X (with respect to the natural uniform convergence structure on F).*

PROOF. Clearly a net, which converges uniformly, also converges in $C_c(X, F)$. Conversely, let $(f_i)_i$ converge to zero in $C_c(X, F)$ and assume that the convergence is not uniform on X. Since F is embedded into $L_c L_c F$, there is a net $(l_k)_k$ in $L_c F$, which converges to some $l \in LF$, such that $l_k \circ f_i(x)$ does not converge to zero uniformly on X. Now there exist an $\varepsilon > 0$, subnets $(l_\kappa)_\kappa$ and $(f_\iota)_\iota$ of the nets $(l_k)_k$ and $(f_i)_i$, respectively, and a net $(x_{\iota\kappa})_{\iota\kappa}$ in X, such that $|l_\kappa \circ f_\iota(x_{\iota\kappa})| \geq \varepsilon$ for every ι and κ. But since $(x_{\iota\kappa})_{\iota\kappa}$ has a convergent subnet, this leads to the contradiction $0 \geq \varepsilon$. Hence $(f_i)_i$ converges to zero uniformly on X.

Next we prove a preparatory result, which will be phrased in terms of cvs although we shall not need it in its full generality in this paper. It must be stressed that it is essential for the proof of Theorem 6, that this result can be obtained for arbitrary quasi-bounded nets in $H_c(U, F)$ and not only for bounded nets.

4. PROPOSITION. *Let U be a circled, convex, and open subset of a cvs E, let F be an L_α-embedded cvs ($\alpha = c$ or e), and let*

$$\gamma \colon H_\alpha(U, F) \to \prod_{m=0}^{\infty} P_\alpha({}^m E, F) \qquad (\alpha = c \text{ or } e)$$

be the continuous linear mapping defined by $\gamma(f) = (\hat{d}^m f(0))_m$. If a net $(f_i)_i$ in $H_\alpha(U, F)$ is quasi-bounded and if the net $(\gamma(f_i))_i$ converges to zero in $\prod_m P_\alpha({}^m E, F)$, then $(f_i)_i$ converges to zero in $H_\alpha(U, F)$ ($\alpha = c$ or e).

PROOF. Let $(f_i)_i$ be a quasi-bounded net in $H_\alpha(U, F)$, such that $\hat{d}^m f_i(0)$ converges to 0 in $P_\alpha({}^m E, F)$ for each $m = 0, 1, \dots$. We shall prove that $f_i \to 0$ in $H_\alpha(U, F)$. Since the mapping $H_\alpha(U, F) \to C_\alpha(U \times L_c F), f \mapsto ((x, l) \mapsto l \circ f(x))$, is an embedding, we have in the case $\alpha = c$ to prove that $l_k \circ f_i(h_j) \to 0$ for each

net $(h_j, l_k)_{jk}$, which converges to $(h, l) \in U \times L_c F$. In the case that $\alpha = e$, we have to prove that there exist j_0 and k_0, such that for each $\varepsilon > 0$ exists an i_0 with $|l_k \circ f_i(h_j)| < \varepsilon$ for $(i, j, k) \geq (i_0, j_0, k_0)$. According to [4]

$$l_k \circ f_i(h_j) = \sum_{m=0}^{N} \frac{l_k \circ \hat{d}^m f_i(0) h_j}{m!} + l_k \circ R_N(f_i, h_j),$$

where

$$R_N(f_i, h_j) = \frac{1}{2\pi i} \int_{|\lambda| = \rho > 1} \frac{f_i(\lambda h_j)}{\lambda^{N+1}(\lambda - 1)} \, d\lambda$$

and where $(h_j, l_k) \to (h, l)$ in $U \times L_c F$. Let $\rho > 1$ be such that $\rho h \in U$. For an arbitrary η with $1 < \eta < \rho$ we have a continuous mapping $L_c F \times H_\alpha(U, F) \times \overline{D}_\eta \times \rho^{-1} U \to \mathbf{C}$, $(l, f, \lambda, h) \mapsto l \circ f(\lambda h)$, where $\overline{D}_\eta = \{\lambda \in \mathbf{C} : |\lambda| \leq \eta\}$. The associated mapping $L_c F \times H_\alpha(U, F) \times \rho^{-1} U \to C_c(D_\eta)$ is continuous by the universal property of continuous convergence [1]. According to Lemma 3 the net $(\lambda \mapsto l_k \circ n^{-1} f_i(\lambda h_j))$ converges to zero uniformly on D_η. Therefore, there are i_0, j_0, k_0, and n_0, such that $\sup_{|\lambda| = \eta} |l_k \circ f_i(\lambda h_j)| \leq n_0$ for $(i, j, k) \geq (i_0, j_0, k_0)$. Since

$$|l_k \circ R_N(f_i, h_j)| \leq 1/\eta^N (\eta - 1) \sup_{|\lambda| = \eta} |l_k \circ f_i(\lambda h_j)|,$$

there is, for a given $\varepsilon > 0$, a number N, such that $|l_k \circ R_N(f_i, h_j)| < \varepsilon/2$ for $(i, j, k) \geq (i_0, j_0, k_0)$ (with the case $\alpha = e$ in mind, observe that i_0, j_0, and k_0 do not depend on ε). For each $m \in \{0, 1, \ldots, N\}$ the net $(l_k \circ \hat{d}^m f_i(0))_{ik}$ converges to zero in $P_\alpha(^m E, F)$ ($\alpha = c$ or e). Thus there is, in the case $\alpha = c$, a triple (i_1, j_1, k_1) with $(i_1, j_1, k_1) \geq (i_0, j_0, k_0)$, such that

$$(1) \qquad \left| \sum_{m=0}^{N} \frac{l_k \circ \hat{d}^m f_i(0) h_j}{m!} \right| < \frac{\varepsilon}{2},$$

so consequently $|l_k \circ f_i(h_j)| < \varepsilon$ for $(i, j, k) \geq (i_1, j_1, k_1)$. Hence $(f_i)_i$ converges to zero in $H_c(U, F)$. In the case $\alpha = e$, there clearly exist j_1 and k_1, such that for each $\varepsilon > 0$ exists an $i_1 \geq i_0$, such that (1) holds for $(i, j, k) \geq (i_1, j_1, k_1)$. Hence $(f_i)_i$ converges to zero in $H_e(U, F)$.

The local nature of the convergence structures c and e immediately gives

5. LEMMA. *If $(U_i)_i$ is an open covering of an open subset U of an lcs E, then the canonical mapping $H_\alpha(U) \to \prod_i H_\alpha(U_i)$ ($\alpha = c$ or e) is an embedding.*

Now we can state

6. THEOREM. *Let U be an open subset of an lcs E. Then E is a Schwartz space if and only if $H_c(U) = H_e(U)$.*

PROOF. Because of Lemma 5 it suffices to prove the theorem for a circled convex U. If $H_c(U) = H_e(U)$ then $L_c E = L_e E$, and consequently E is Schwartz [11]. Conversely, if E is Schwartz, then $P_c(^m E) = P_e(^m E)$ for each $m \in \mathbf{N}$ (cf. [5, Proposition 2.3]). Consider an arbitrary net $(f_i)_i$, which converges to zero in $H_c(U)$. Then $(f_i)_i$ is quasi-bounded in $H_c(U)$ and $(\hat{d}^m f_i(0)/m!)_i$ converges to zero in $P_c(^m E) = P_e(^m E)$ for each $m = 0, 1, \ldots$. Since $(f_i)_i$ is quasi-bounded also

in $H_e(U)$ it follows by Proposition 4 that $(f_i)_i$ converges to zero in $H_e(U)$. Thus $H_c(U) = H_e(U)$.

REMARK. If one could prove that $H_c(U)$ (or $H_e(U)$) is a Schwartz cvs [12] when U is an open subset of a Schwartz lcs, then one would have an equality $H_c(U, F) = H_e(U, F)$ for any Schwartz lcs F.

6.1. COROLLARY. *If U is an open subset of a Schwartz lcs E, then all bounded sets in $H_e(U)$ are relatively compact in $H_e(U)$.*

Every cvs E defines an lcs E_τ, which is the same vector space provided with the finest locally convex topology coarser than the convergence structure of E. It is known that $C_c(X)_\tau = C_{co}(X)$, if X is a c-embedded space (cf. [1, Theorem 37]). On the dual LE of an lcs E, $L_{co}\hat{E} = L_c E_\tau$ (cf. [10, Theorem 9.3.7]), where $\hat{E}$ denotes the completion of E. These two results suggest that a similar result could be valid for spaces of holomorphic functions. But the following example shows, that the situation is completely different in the holomorphic case.

EXAMPLE. Consider the topological product $E = \mathbf{C}^{\mathbf{N}}$. The space E is a nuclear Fréchet space. Hence $H_{co}(E) = H_{\tau_\omega}(E)$ and $H_{co}(E) \neq H_{\tau_\delta}(E)$ (cf. Corollary 6.40 and example 2.52 in [7]). Since every nuclear Fréchet space is separable, $H_e(E)_\tau = H_{\tau_\delta}(E)$ by [4, Corollary 3.3]. But since $H_c(E) = H_e(E)$ by Theorem 6, we can conclude that $H_c(E)_\tau = H_e(E)_\tau = H_{\tau_\delta}(E) \neq H_{co}(E) = H_{\tau_\omega}(E)$.

QUESTION. It would be interesting to know whether the equality $H_c(E)_\tau = H_{co}(E)$ holds or not for a Banach space E.

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