THE BLOCH SPACE AND BMO ANALYTIC FUNCTIONS IN THE TUBE OVER THE SPHERICAL CONE

DAVID BEKOLLE

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ABSTRACT. We prove that the Bloch space coincides with the space BMOA in the tube over the spherical cone of \( \mathbb{R}^3 \); this extends a well-known one-dimensional result.

Introduction. Let \( \Omega \) be a symmetric Siegel domain of type II contained in \( \mathbb{C}^n \). Let \( V \) denote the Lebesgue measure in \( \Omega \) and \( H(\Omega) \) the space of holomorphic (or analytic) functions in \( \Omega \).

When \( n = 1 \) and \( \Omega = \pi^+ = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \), a Bloch function is an element \( f \) of \( H(\pi^+) \) which satisfies the estimate
\[
\|f\|_{\mathcal{B}} = \sup_{z=x+iy \in \pi^+} \{y|f'(z)|\} < \infty.
\]
The Bloch space \( \mathcal{B} \) of \( \pi^+ \) is then the quotient space of the space of Bloch functions by the subspace of constant functions.

It is well known that in \( \pi^+ \), the Bloch space \( \mathcal{B} \) coincides with the quotient space BMOA of the space of BMO analytic functions by the subspace of constant functions. The definition of BMO in \( \pi^+ \) is the same as that of (solid) BMO in the unit disk (cf. [6, p. 631]): in \( \pi^+ \), a locally integrable function \( f \) is said to be BMO if there exists a constant \( C \) such that for any disk \( D \) contained in \( \pi^+ \), there is a constant \( f_D \) such that
\[
\frac{1}{|D|} \int_D |f - f_D| dV \leq C.
\]
In \( \mathbb{C}^2 \), this result can easily be extended to the cartesian product \( (\pi^+)^2 \) of two upper half-planes. In this case, a Bloch function is an element of \( H[(\pi^+)^2] \) which satisfies the estimate
\[
\|f\|_{\mathcal{B}} = \sup_{z=(x_0,x_1) \in (\pi^+)^2} \{y_0y_1 \left| \frac{\partial^2}{\partial x_0 \partial x_1} f(z) \right| \} < \infty.
\]
The Bloch space \( \mathcal{B} \) of \( (\pi^+)^2 \) is then the quotient space of the space of Bloch functions by the subspace
\[
\mathcal{N} = \left\{ f \in H[(\pi^+)^2] : \frac{\partial^2}{\partial x_0 \partial x_1} f(z) \equiv 0 \right\}.
\]
On the other hand, a holomorphic function \( f \) in \( (\pi^+)^2 \) is said to be BMO if there exists a constant \( C \) such that for any bidisk \( D \) contained in \( (\pi^+)^2 \), there is a function \( f_D \in \mathcal{N} \) such that

\[
\frac{1}{|D|} \int_D |f - f_D| \, dV \leq C.
\]

The space BMOA of \( (\pi^+)^2 \) is then the quotient space of the space of BMO analytic functions by the subspace \( \mathcal{N} \). The referee has pointed out to the author that there is a danger of confusion between our space BMOA of \( (\pi^+)^2 \) and the so-called space BMO\((\mathbb{R}^4_+ \times \mathbb{R}^4_+)\) that shows up as the dual space of the two-parameter Hardy space \( H^1(\mathbb{R}^4_+ \times \mathbb{R}^4_+) \) (cf. [4]); the two spaces, however, have nothing to do with each other.

The purpose of this paper is to extend this identity between the Bloch space and the space BMOA to the tube \( \Omega \) over the spherical cone \( \Gamma \) of \( \mathbb{R}^3 \) defined by

\[
\Gamma = \{(y_0, y_1, y_2) \in \mathbb{R}^3 : y_0 y_1 - y_0^2 < 0, \ y_0 > 0\}.
\]

The Bloch space \( \mathcal{B} \) of \( \Omega \) is defined in [2] in terms of the differential operator

\[
\Box_z = 4 \frac{\partial^2}{\partial z_0 \partial z_1} - \frac{\partial^2}{\partial z_2^2}, \quad z = (z_0, z_1, z_2) \in \mathbb{C}^3;
\]

\( \Box \) is the wave operator in \( \mathbb{C}^3 \). More precisely, if \( B(\zeta, z) \) denotes the Bergman kernel of \( \Omega \), a Bloch function in \( \Omega \) is an element \( f \) of \( H(\Omega) \) which satisfies the estimate

\[
\|f\|_{\mathcal{B}} = \sup_{z \in \Omega} \{B^{-1/3}(z, z) |\Box f(z)|\} < \infty.
\]

The Bloch space \( \mathcal{B} \) of \( \Omega \) is then the quotient space of the space of Bloch functions by the subspace \( \mathcal{N} = \{f \in H(\Omega) : \Box f = 0\} \).

The problem is actually to give a good definition of the space BMOA of \( \Omega \); this consists of the description of a suitable family of geometric figures that will play the same role as the disks in the upper half-plane \( \pi^+ \) and the bidisks in the product \( (\pi^+)^2 \) of two upper half-planes.

In §1, we recall some results of [1 and 2] about the Bloch space in the tube \( \Omega \) over the spherical cone. According to one of these results, the Bloch space \( \mathcal{B} \) of \( \Omega \) can be realized as the Bergman projection of \( L^\infty \). Let us mention that, since the Bergman kernel \( B(\zeta, z) \) of \( \Omega \) is not integrable with respect to \( z \), we have to state the definition of the Bergman projection \( Pb \) of a bounded function \( b \) and we shall also recall the expression of an integral kernel which defines the operator \( P \) in \( L^\infty \).

When we prove that every Bloch function \( f \) is BMO, this kernel will be used in the definition of the function of \( \mathcal{N} \) which will be subtracted from \( f \).

In §2, we define the space BMOA of \( \Omega \) by describing a suitable family \( \mathcal{G} \) of geometric figures: up to an affine change of coordinates, the elements of \( \mathcal{G} \) are polydisks (cartesian products of three disks).

In §3, we prove the identity between the Bloch space \( \mathcal{B} \) and the space BMOA of \( \Omega \). Let us note that this identity can be extended with the same proof, by means of the results of [3, 2 and 1] to the following domains: the Cayley transform of the unit ball in \( \mathbb{C}^n \), \( n \geq 2 \); the tube over the spherical cone of \( \mathbb{R}^n \), \( n \geq 4 \); and the tube over the cone of real symmetric positive definite \( m \times m \) matrices, \( m \geq 3 \).

The author expresses his sincere thanks to R. R. Coifman for raising this problem and to R. Rochberg for suggesting that the family should be invariant under the affine automorphisms of \( \Omega \).
1. Preliminary results about the Bloch space. In this paragraph, we recall some results proved in [1 and 2]. In the following, $\Omega$ will denote the tube $\mathbb{R}^3 + i\Gamma \subset \mathbb{C}^3$ over the spherical cone $\Gamma$ of $\mathbb{R}^3$. The Bergman kernel $B(\xi, z)$ of $\Omega$ has the following expression:

$$B(\xi, z) = c[(\xi_0 - \bar{z}_0)(\xi_1 - \bar{z}_1) - (\xi_2 - \bar{z}_2)^2]^{-3},$$

where $\xi = (\xi_0, \xi_1, \xi_2)$ and $z = (z_0, z_1, z_2)$ are two points of $\Omega$.

The Bloch space $\mathcal{B}$ of $\Omega$ is defined in the introduction; equipped with the norm induced by the seminorm

$$||f||_{\mathcal{B}} = \sup_{z \in \Omega} \{B^{-1/3}(z, z)|\Box f(z)|\}$$

defined in the space of Bloch functions, the Bloch space $\mathcal{B}$ is a Banach space.

In [2] (cf. also [1]), we extended to $\Omega$ a well-known one-dimensional result: the Bloch space can be realized as the Bergman projection of $L^\infty$. In fact, let us first recall that by Bergman projection, one originally means the orthogonal projection $P$ of $L^2(dV)$ onto the Bergman space

$$A^2(\Omega) = L^2(dV) \cap H(\Omega),$$

defined on $L^2(dV)$ by

$$P\phi(\zeta) = \int_\Omega B(\zeta, z)\phi(z) dV(z), \quad \zeta \in \Omega.$$
THEOREM 1.1. The Bergman projection $P$, which assigns to a bounded function $b$ the element $Pb$ of $\mathcal{B}$ represented by the Bloch function $g$ defined by (2), is a continuous operator from $L^\infty$ onto the Bloch space $\mathcal{B}$ of $\Omega$. Furthermore, the projection $P$ possesses a continuous right inverse $R: \mathcal{B} \to L^\infty$.

Let us mention the following lemma, which was used in the proof of the preceding theorem.

LEMMA 1.1. Let $K$ be a compact subset of $\Omega$. There exist a constant $C = C(K)$ and a positive function $M \in L^1(dV)$ such that for any $\zeta \in K$ and any $z \in \Omega$, the following inequality holds:

$$|(B - B_0)(\zeta, z)| \leq CM(z).$$

2. The space $\text{BMOA}$. In the introduction, we reduced the definition of the space $\text{BMOA}$ to the description of a suitable family $\mathcal{F}$ of geometric figures.

Let us then describe such a family $\mathcal{F}$. We first describe a subfamily $\mathcal{F}_0$ of $\mathcal{F}$; $\mathcal{F}_0$ will consist of all polydisks centered at $e = (i, i, 0)$, whose multiradius $(R_0, R_1, R_2)$ satisfies $0 < R_j \leq 1/4$ for $j = 0, 1, 2$. Notice that those polydisks are all contained in $\Omega$.

Now, let $\text{Aff}(\Omega)$ denote the simply transitive group of affine automorphisms of $\Omega$ defined in the first paragraph of [7]; the whole family $\mathcal{F}$ is defined by

$$\mathcal{F} = \bigcup_{\phi \in \text{Aff}(\Omega)} \phi(\mathcal{F}_0).$$

Let us describe more precisely the figures of $\mathcal{F}$. We shall represent the domain $\Omega$ by the space of symmetric $2 \times 2$ matrices in the following way: a point $z = (z_0, z_1, z_2)$ of $\Omega$ will take the form

$$z = \begin{pmatrix} z_0 & z_2 \\ z_2 & z_1 \end{pmatrix}.$$
Let $Q$ be a figure of $\mathcal{G}$, $Q = \phi(D)$, where $D$ is an element of $\mathcal{G}_0$ whose multiradius is denoted by $(R_0, R_1, R_2)$. A point $z = (z_0, z_1, z_2)$ belongs to $Q$ if

$$
\begin{align*}
|z_1 - \zeta_1| &< \chi_2(t)R_1, \\
|z_2 - \zeta_2 - \frac{t_2}{\chi_2(t)}(z_1 - \zeta_1)| &< [\chi_1(t)\chi_2(t)]^{1/2}R_2, \\
|z_0 - \zeta_0 - \frac{2t_2}{\chi_2(t)}(z_2 - \zeta_2) + \left(\frac{t_2}{\chi_2(t)}\right)^2(z_1 - \zeta_1)| &< \chi_1(t)R_0.
\end{align*}
$$

Now, by means of the affine change of coordinates defined by

$$
\begin{align*}
&z_1' = z_1, \\
&z_2' = z_2 - \frac{t_2}{\chi_2(t)}(z_1 - \zeta_1), \\
&z_0' = z_0 - \frac{2t_2}{\chi_2(t)}(z_2 - \zeta_2) + \left(\frac{t_2}{\chi_2(t)}\right)^2(z_1 - \zeta_1),
\end{align*}
$$

we obtain that $Q$ is the polydisk centered at $\zeta = (\zeta_0, \zeta_1, \zeta_2)$, defined by

$$Q = \{(z_0', z_1', z_2') \in C^3 : |z_1' - \zeta_1| < \chi_2(t)R_1, |z_2' - \zeta_2| < [\chi_1(t)\chi_2(t)]^{1/2}R_2, |z_0' - \zeta_0| < \chi_1(t)R_0\}.$$

Let us next define the space $\text{BMOA}$ of $\Omega$. A function $f \in H(\Omega)$ is said to be $\text{BMO}$ if there exists a constant $C$ such that for any $Q \in \mathcal{G}$, there is a function $f_Q \in \mathcal{N}$, where $\mathcal{N} = \{f \in H(\Omega) : \square f = 0\}$, such that

$$\frac{1}{|Q|} \int_Q |f - f_Q| \, dV \leq C.$$

Let $\|f\|_{\text{BMOA}}$ denote the smallest constant for which this property holds. As a definition, the space $\text{BMOA}$ of $\Omega$ is the quotient space of the $\text{BMO}$ analytic functions by the subspace $\mathcal{N}$; equipped with the norm induced by $\|\|_{\text{BMOA}}$, the space $\text{BMOA}$ is a Banach space.

### 3. The identity between the Bloch space $\mathcal{B}$ and the space $\text{BMOA}$.

We shall now prove the following theorem.

**THEOREM 3.1.** In the tube $\Omega$ over the spherical cone $\Gamma$ of $R^3$, the Bloch space $\mathcal{B}$ coincides with the space $\text{BMOA}$ and the norms induced on $\mathcal{B}$ by $\|\|_\mathcal{B}$ and on $\text{BMOA}$ by $\|\|_{\text{BMOA}}$ are equivalent.

**PROOF.** We first show that there exists a constant $C$ such that any Bloch function $f$ satisfies $\|f\|_{\text{BMOA}} \leq C\|f\|_{\mathcal{B}}$. This will immediately yield the inclusion of $\mathcal{B}$ in $\text{BMOA}$.

Let then $f$ be a Bloch function; in the following, $B_0$ denotes the kernel defined in [1]. In view of Theorem 1.2, there exists a bounded function $b$ such that the following equality holds, modulo an element of $\mathcal{N}$:

$$f(\zeta) = \int_{\Omega} (B - B_0)(\zeta, z)b(z) \, dV(z),$$

i.e. $f$ belongs to the same equivalence class of $\mathcal{B}$ as the function in the right-hand side of (3).
We must prove that there exists a constant \( C \) such that for any Bloch function \( f \) and any polydisk \( Q \in \mathcal{S} \), there is a function \( f_Q \in \mathcal{N} \) such that

\[
\frac{1}{|Q|} \int_Q |f - f_Q| \, dV \leq C\|f\|_\mathcal{S}.
\]

Let \( c \) denote the center of \( Q \) and let \( \phi \) be the element of \( \text{Aff}(\Omega) \) that assigns \( c \) to \( e \). Define the function \( f_Q \) by

\[
(f - f_Q)(\zeta) = \int_{\Omega} (B - B_0)(\phi(\zeta), \phi(z)) b(z) |J\phi|^2 \, dV(z),
\]

where \( J\phi \) denotes the jacobian of \( \phi \).

Let \( \mathcal{C} \) denote the center of \( Q \) and let \( \phi \) be the element of \( \text{Aff}(\Omega) \) that assigns \( c \) to \( e \). Define the function \( f_Q \) by

\[
(f - f_Q)(\zeta) = \int_{\Omega} (B - B_0)(\phi(\zeta), \phi(z)) b(z) |J\phi|^2 \, dV(z),
\]

where \( J\phi \) denotes the jacobian of \( \phi \).

Let us verify that the so-defined function \( f_Q \) belongs to \( \mathcal{N} \); in view of Lemma 1.1, (3) and the relation \( B(\phi(\zeta), \phi(z))|J\phi|^2 = B(\zeta, z) \), it suffices to check that

\[
\Box \{ B_0(\phi(\zeta), \phi(z))|J\phi|^2 - B_0(\zeta, z) \} \equiv 0,
\]

and since \( \Box B_0(\zeta, z) \equiv 0 \) and \( J\phi \) is independent of \( \zeta \), this is equivalent to

\[
\Box B_0(\phi(\zeta), \phi(z)) \equiv 0.
\]

This verification then follows from the following lemma, whose proof is elementary.

**Lemma 3.1.** For any \( f \in H(\Omega) \) and any \( \phi \in \text{Aff}(\Omega) \), the following equality holds.

\[
\Box (f \circ \phi) = |J\phi|^{2/3} \cdot (\Box f) \circ \phi.
\]

Let us next prove that the function \( f_Q \) defined in (5) gives rise to inequality (4). In view of (5) and the Fubini theorem, we get

\[
\frac{1}{|Q|} \int_Q |(f - f_Q)(\zeta)| \, dV(\zeta) \leq \int_{\Omega} N(z) b(z) \, dV(z) \leq \|b\|_\infty \int_{\Omega} N(z) \, dV(z),
\]

where

\[
N(z) = \frac{1}{|Q|} \int_Q |(B - B_0)(\phi(\zeta), \phi(z))| |J\phi|^2 \, dV(\zeta).
\]

Now, in view of Theorem 1.1, if \( R \) denotes the continuous right inverse of \( P : L^\infty \to \mathcal{S} \), we can take \( b = Rf \) and this yields the estimate \( \|b\|_\infty \leq C\|f\|_\mathcal{S} \).

It then suffices to show that there exists a constant \( C' \) such that for any \( Q \in \mathcal{S} \), the function \( N \) defined in (6) satisfies the estimate

\[
\int_{\Omega} N(z) \, dV(z) \leq C'.
\]

In order to prove (7), we apply to (6) the change of variables \( \zeta' = \phi(\zeta) \) and \( z' = \phi(z) \) and we let \( D \) denote the polydisk \( D = \phi^{-1}(Q) \) centered at \( e \) (\( D \) belongs to the subfamily \( \mathcal{S}_0 \) of \( \mathcal{S} \)); since \( |Q| = |J\phi|^{-2} |D| \), it follows from the Fubini theorem that the left-hand side of (7) is equal to \( (1/|D|) \int_D \{ \int_{\Omega} |(B - B_0)(\zeta', z')| \, dV(z') \} \, dV(z') \).

Applying Lemma 1.1 to the compact subset

\[
\bar{D}_0 = \{(z_0, z_1, z_2) \in \mathbb{C}^3 : |z_0 - i| \leq 1/4, |z_1 - i| \leq 1/4, |z_2| \leq 1/4\}
\]

immediately yields inequality (7).
We now prove the converse implication, i.e. there exists a constant $C$ such that any BMO analytic function $f$ satisfies the estimate $\|f\|_\mathcal{A} \leq C\|f\|_{\text{BMOA}}$.

The method of proof is the same as that of R. R. Coifman, R. Rochberg and G. Weiss [6] for the same property in the unit disk of the complex plane.

We must show that there exists a constant $C$ such that for any BMO analytic function $f$ and any $\zeta$ in $\Omega$, the following inequality holds:

\begin{equation}
B^{-1/3}(\zeta, \zeta) |\Box f(\zeta)| \leq C\|f\|_{\text{BMOA}}.
\end{equation}

Let us first prove inequality (8) for any $\zeta$ in the polydisk $D$ centered at $e$, whose multiradius is $(1/5, 1/5, 1/5)$. We let $D_0$ denote the polydisk centered at $e$, whose multiradius is $(1/4, 1/4, 1/4)$ and we let $f_0$ denote an element of $\mathcal{N}$ such that

$$\frac{1}{|D_0|} \int_{D_0} |f - f_0| \ dV \leq \|f\|_{\text{BMOA}}.$$

We use the following lemma.

**Lemma 3.2.** If $g \in H(\Omega)$ satisfies $\left(\int_{D_0} \frac{1}{D_0} \int_{D_0} |g| \ dV\right) \leq C < \infty$, then for any $\zeta$ in $D$, the inequality $|\Box g(\zeta)| \leq 135C$ holds.

**Proof of Lemma 3.2.** In view of Cauchy’s formula for polydisks, the following equality holds for any $\zeta$ in $D$:

$$\left(\frac{400}{9\pi}\right)^3 \int_{r_2=1/5}^{1/4} \int_{r_1=1/5}^{1/4} \int_{r_0=1/5}^{1/4} \left( \int_{\partial D_r} \frac{g(z) \ dz_0 \wedge dz_1 \wedge dz_2}{(z_0 - \zeta)^2(z_1 - \zeta)^2(z_2 - \zeta)^2} \right) \ dr_0 \ dr_1 \ dr_2,
$$

where

$$\partial D_r = \{(z_0, z_1, z_2) \in \mathbb{C}^3 : |z_0 - i| = r_0, |z_1 - i| = r_1, |z_2| = r_2\}.$$

We then easily obtain that

$$\left| \frac{\partial^2}{\partial \zeta_0 \partial \zeta_1} g(\zeta) \right| \leq \left(\frac{25}{9}\right)^3 \frac{1}{|D_0|} \int_{D_0} |g| \ dV$$

and the same estimate also holds for $|\partial^2/\partial \zeta_2^2 g(\zeta)|$; the conclusion immediately follows from the hypothesis.

Now, in view of Lemma 3.2 and of the equality $\Box f_0 = 0$, inequality (8) when $\zeta$ lies in $D$ is a consequence of the following lemma, whose proof is easy.

**Lemma 3.3.** There exist two positive constants $C_1$ and $C_2$ such that for any $\zeta$ in $D$, $C_1 \leq B(\zeta, \zeta) \leq C_2$.

Let us next prove inequality (8) when $\zeta$ is any point in $\Omega$. We let $D$ again denote the polydisk centered at $e$, whose multiradius is $(1/5, 1/5, 1/5)$; since the family $\{\phi(D) : \phi \in \text{Aff}(\Omega)\}$ is a covering of $\Omega$, it is enough to prove that for any $\phi \in \text{Aff}(\Omega)$, inequality (8) holds for any $\zeta$ in $\phi(D)$.

Let $Q = \phi(D)$ and $Q_0 = \phi(D_0)$, where $D_0$ again denotes the polydisk centered at $e$ whose multiradius is $(1/4, 1/4, 1/4)$. Let $f_\phi$ be an element of $\mathcal{N}$ such that

\begin{equation}
\frac{1}{|Q_0|} \int_{Q_0} |(f - f_\phi)(z)| \ dV(z) \leq \|f\|_{\text{BMOA}}.
\end{equation}
Applying the change of variable $z = \phi(z')$ to the left-hand side of (9) yields
\[
\frac{1}{|D_0|} \int_{D_0} |(f - f_\phi) \circ \phi(z')| \, dV(z') \leq \|f\|_{\text{BMOA}}.
\]
Now, in view of Lemma 3.3, we get for any $\zeta'$ in $D$:
\[
|\Box_{\zeta'}[(f - f_\phi) \circ \phi](\zeta')| \leq 135\|f\|_{\text{BMOA}}.
\]
We next apply the change of variable $\zeta = \phi(\zeta')$ to the left-hand side of this inequality; in view of Lemma 3.1 and of the equality $\Box f_\phi = 0$, this yields
\[
|\Box_{\zeta}f(\zeta)||J\phi|^{2/3} \leq 135\|f\|_{\text{BMOA}}.
\]
The following lemma then completes the proof of the theorem.

**Lemma 3.4.** $D$ denotes the polydisk centered at $e$, whose multiradius is $(1/5, 1/5, 1/5)$. There exist two constants $C_1$ and $C_2$ such that for any $\phi \in \text{Aff}(\Omega)$ and any $\zeta \in \phi(D)$,
\[
C_1|J\phi|^{-2} \leq B(\zeta, \zeta) \leq C_2|J\phi|^{-2}.
\]

**Proof of Lemma 3.4.** This lemma is a straightforward consequence of Lemma 3.2 and of the relation $B(\zeta, \zeta) = B(\zeta', \zeta')|J\phi|^{-2}$, where $\zeta = \phi(\zeta')$.

**References**


**Département de Mathématiques, Université de Yaoundé, BP. 812, Yaoundé, Cameroun**