THE GAP BETWEEN $\text{cmp} X$ AND $\text{def} X$ CAN BE ARBITRARILY LARGE

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Dedicated to Professor Yukihiro Kodama on his 60th birthday

ABSTRACT. We give an example of a separable metrizable space $X$ with $\text{def} X - \text{cmp} X = n$ for every $n \in \mathbb{N}$.

1. Introduction. In this paper all spaces are separable and metrizable.

The compactness degree, $\text{cmp} X$, of a space $X$ is defined as follows: a space $X$ satisfies $\text{cmp} X = -1$ if $X$ is compact; if $n$ is a nonnegative integer, then $\text{cmp} X \leq n$ means that each point of $X$ has arbitrarily small neighborhoods $U$ with $\text{cmp} \text{Bd} U \leq n - 1$. We put $\text{cmp} X = n$ if $\text{cmp} X \leq n$ and $\text{cmp} X \notin n - 1$. If there is no integer $n$ for which $\text{cmp} X \leq n$, then we put $\text{cmp} X = \infty$.

The compactness deficiency, $\text{def} X$, of a space $X$ is the least integer $n$ for which $X$ has a compactification $\alpha X$ with $\dim(\alpha X - X) < n$. We allow $n$ to be $\infty$.

In general, the inequality $\text{cmp} X \leq \text{def} X$ holds. The well-known conjecture of J. de Groot (see [2]) that $\text{cmp} X = \text{def} X$ has been negatively solved by R. Pol [5]; the space $X$ of Pol’s example has $\text{cmp} X = 1$ and $\text{def} X = 2$. In the review of R. Pol’s paper [5], J. van Mill [3] states “It seems still to be open whether the gap between $\text{cmp} X$ and $\text{def} X$ can be arbitrarily large.”

The purpose of this paper is to answer this question affirmatively. Namely, we shall give the following example.

EXAMPLE. For every $n \in \mathbb{N}$ there exists a space $X$ such that $\text{def} X - \text{cmp} X = n$.

2. Preliminaries. Let $S$ be a collection of subsets of a space $X$. Then we shall write $[S]^n$ for $\{T: T \subseteq S \text{ with } |T| = n\}$, $\text{Bd} S$ for $\{\text{Bd} S: S \in S\}$ and $\bigcap S$ for $\bigcap\{S: S \in S\}$.

Let $Y$ be a subspace of a space $X$ and $U$ a collection of open subsets of $X$. Then $U$ is an outer base for $Y$ in $X$ if for every $y \in Y$ and any neighborhood $V$ of $y$ in $X$ there is $U \in U$ such that $y \in U \subseteq V$.

The following lemma is needed in §4; the proof is straightforward.

2.1. LEMMA. Let $X$ be a space with $\text{def} X < n$ and $\{(E_j, F_j): 1 \leq j \leq n\}$ a collection of pairs of disjoint compact subsets of $X$. Then for each $j$, $1 \leq j \leq n$, there is a partition $T_j$ in $X$ between $E_j$ and $F_j$ such that $\bigcap\{T_j: 1 \leq j \leq n\}$ is compact.

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To show our example it suffices to construct a space \( Y \) with \( n \leq \text{def } Y - \text{cmp } Y < \infty \). Indeed, for this space \( Y \) we construct another space \( Z \) with \( \text{cmp } Z = \text{def } Z = \text{def } Y - n \); such a space exists (see [2, Theorem 3.1.1]). Let \( X = Y \oplus Z \) be the topological sum of \( Y \) and \( Z \). Then

\[
\text{cmp } X = \max\{\text{cmp } Y, \text{cmp } Z\} = \text{cmp } Z = \text{def } Y - n
\]

and

\[
\text{def } X = \max\{\text{def } Y, \text{def } Z\} = \text{def } Y.
\]

Thus we have \( \text{def } X - \text{cmp } X = n \).

In the next section we shall construct a space \( X \) such that \( m \leq \text{def } X - \text{cmp } X \leq 2m \) for every \( m \in \mathbb{N} \).

Throughout the rest of this paper, we shall fix a positive integer \( m \) and put \( n = 2m + 1 \). Let \( I = [0,1] \) be the closed unit interval.

3. Construction. Let

\[
\partial I^n = \{(x_j) \in I^n : x_j = 0 \text{ or } 1 \text{ for some } j, 1 \leq j \leq n\}
\]

be the combinatorial boundary of the \( n \)-dimensional cube \( I^n \). We take countable, dense subsets \( D_0 \) and \( D_1 \) in \((0,1)\) with \( D_0 \cap D_1 = \emptyset \). Let us set

\[
M_i = \{(x_j) \in (0,1)^n : |\{j : x_j \in D_i\}| \geq n - m\},
\]

and

\[
L_i = (0,1)^n - M_i
\]

for each \( i = 0,1 \). Then, obviously, \( M_0 \cap M_1 = \emptyset \) and by [1, 1,8.5], \( \dim L_i = m \).

Then, by [4, 12.12–13], there are two collections \( \mathcal{B}_0 \) and \( \mathcal{B}_1 \) of open subsets of \( I^n \) satisfying the following conditions (1) to (6) below:

1. \( \mathcal{B}_0 \) is an outer base for \((I^{n-1} \times [0, \frac{1}{2}]) \cap \partial I^n \) in \( I^n \),
2. \( \mathcal{B}_1 \) is an outer base for \((I^{n-1} \times (\frac{1}{2}, 1]) \cap \partial I^n \) in \( I^n \),
3. \( \mathcal{F} \cap L_i = \emptyset \) for every \( \mathcal{F} \in [\partial I_i]^{m+1} \) and each \( i = 0,1 \),
4. \( \text{Cl } B \subset I^{n-1} \times [0, \frac{2}{3}) \) for every \( B \in \mathcal{B}_0 \),
5. \( \text{Cl } B \subset I^{n-1} \times (\frac{1}{3}, 1] \) for every \( B \in \mathcal{B}_1 \), and
6. \( |\mathcal{B}_i| = \omega \) for each \( i = 0,1 \).

By (6), \( [\partial I_i]^{m+1} \) is countable; therefore, we enumerate it as \( [\partial I_i]^{m+1} = \{\mathcal{F}_j : j \in \mathbb{N}\} \). Let us see \( F_{ij} = \bigcap \mathcal{F}_j \), and let

\[
E_{ik} = \bigcup\{F_{ij} : j \leq k\} - \partial I^n
\]

for \( i = 0,1 \) and \( k \in \mathbb{N} \). Then, by (3), we have \( E_{0k} \cap E_{1k} \subset M_0 \cap M_1 = \emptyset \). Thus \( E_{0k} \) and \( E_{1k} \) are disjoint closed subsets of \((0,1)^n\), therefore we can take disjoint open subsets \( U_{0k} \) and \( U_{1k} \) in \((0,1)^n\) such that

7. \( E_{ik} \subset U_{ik} \) for each \( i = 0,1 \),
8. \( U_{0k} \subset I^{n-1} \times [0, \frac{2}{3}) \), and
9. \( U_{1k} \subset I^{n-1} \times (\frac{1}{3}, 1] \).

Let us set

\[
X_k = (I^n - U_{0k} \cup U_{1k}) \times \{1/k\} \text{ for every } k \in \mathbb{N},
\]

\[
X_0 = \partial I^n \times \{0\}, \text{ and}
\]

\[
X = \bigcup\{X_k : k = 0,1,2,\ldots\}.
\]
We regard $X$ as the subspace of the $(n + 1)$-dimensional cube

$$I^{n+1} = \prod \{I_j : 1 \leq j \leq n + 1\},$$

where $I_j$ is the copy of $I$.

4. $2m \leq \text{def } X \leq 2m + 1$. Note that def $Y \leq \dim Y$ for every space $Y$ (see [5, Theorem 2.1.1]). Since $\dim X \leq n = 2m + 1$, we have def $X \leq 2m + 1$. Assume that def $X < 2m = n - 1$. Let us set

$$J_j = (I_1 \times \cdots \times I_{j-1} \times \{0\} \times I_{j+1} \times \cdots \times I_{n+1}) \cap X,$$

and

$$K_j = (I_1 \times \cdots \times I_{j-1} \times \{1\} \times I_{j+1} \times \cdots \times I_{n+1}) \cap X$$

for every $j$, $1 \leq j \leq n - 1$. Then $J_j$ and $K_j$ are disjoint compact subsets of $X$. Thus, by Lemma 2.1, there is a partition $T_j$ in $X$ between $J_j$ and $K_j$ for every $j$, $1 \leq j \leq n - 1$, such that $\bigcap\{T_j : 1 \leq j \leq n - 1\}$ is compact. Since $T_j \cap X_k$ is a partition in $X_k$ between $J_j \cap X_k$ and $K_j \cap X_k$, and $X_k$ is closed in $I^n \times \{1/k\}$, there is a partition $T_{jk}$ in $I^n \times \{1/k\}$ between $J_j \cap X_k$ and $K_j \cap X_k$ such that $T_{jk} \cap X_k \subset T_j \cap X_k$ for each $j$, $1 \leq j \leq n - 1$, and each $k \in \mathbb{N}$. Let $S_k$ be a continuum meeting $I^{n-1} \times \{1/6\} \times \{k/6\}$ and $I^{n-1} \times \{5/6\} \times \{1/k\}$ in $I^n \times \{1/k\}$ with $S_k \subset \bigcap\{T_{jk} : 1 \leq j \leq n - 1\} \cap (I^{n-1} \times [1/6, 5/6] \times \{1/k\})$ (see [6, Lemma 5.2]). Since $S_k$ is connected, by (8) and (9), we have $S_k \not\subset U_0 \cup U_1$. Thus we have $S_k \cap X_k \neq \emptyset$ for every $k \in \mathbb{N}$. Obviously, $S_k \cap X_k \subset \bigcap\{T_j : 1 \leq j \leq n - 1\} \cap X_k \subset \bigcap\{T_j : 1 \leq j \leq n - 1\}$ and $\{S_k \cap X_k : k \in \mathbb{N}\}$ is discrete in $X$. This contradicts the compactness of $\bigcap\{T_j : 1 \leq j \leq n - 1\}$. Hence we have def $X \geq n - 1 = 2m$.

5. $1 \leq \text{cmp } X \leq m$. Note that cmp $X \leq 0$ if and only if def $X \leq 0$ (see [2, Main Theorem]). Since def $X \geq 2m > 0$, we have cmp $X \geq 1$.

We shall prove that cmp $X \leq m$. To prove this we only consider points of $X_0$, because $\bigcup\{X_k : k \in \mathbb{N}\}$ is locally compact and open in $X$. First we shall show the following

Claim. Let $1 \leq l \leq m$. For every $\{B_1, \ldots, B_l\} \subset [I]^l$ and any $(k_1, \ldots, k_l) \in \mathbb{N}^l$ we have cmp $\cap\{B_{d_X} B_j' : 1 \leq j \leq l\} \leq m - l$, where $B_j' = (B_j \times [0,1/k_j]) \cap X$ for each $j$, $1 \leq j \leq l$.

Proof of Claim. We proceed by downward induction on $l$.

Step 1. $l = m$. Let $Y = \bigcap\{B_{d_X} B_j' : 1 \leq j \leq m\}$, $y \in Y$, and $U$ be a neighborhood of $y$ in $Y$. We show that there is a neighborhood $V$ of $y$ in $Y$ such that $V \subset U$ and $B_{d_Y} V$ is compact. We may assume that $y \in X_0$. Then, by (1), (2), (4) and (5), there are $B_{m+1} \subset B_i$ and $k \in \mathbb{N}$ such that $y \in (B_{m+1} \times [0,1/k]) \cap Y \subset U$. Since $\{B_1, \ldots, B_m, B_{m+1}\} \subset [I]^{m+1}$, $\bigcap\{B_{d_Y} B_j : 1 \leq j \leq m + 1\} = F_{ip}$ for some $p \in \mathbb{N}$. Let $V = (B_{m+1} \times [0,1/q]) \cap Y$, where $q = \max\{k, p\}$. Then $V$ is a neighborhood of $y$ in $Y$. Obviously, we have $V \subset U$. By (7), it is easy to see that

$$B_{d_Y} V \subset \left(\bigcap\{B_{d_Y} B_j : 1 \leq j \leq m + 1\} \cap \partial I^n\right) \times \{0,1/(p+1),1/(p+2),\ldots\} \subset X.$$

Hence $B_{d_Y} V$ is compact; therefore, we have cmp $Y \leq 0 = m - l$.

Step 2. Let $1 \leq l < m$ and suppose that the Claim is satisfied for $l + 1$. Therefore, we get cmp $\cap\{B_{d_X} B_j' : 1 \leq j \leq l\} \leq m - (l + 1)$.
Let \( Y = \bigcap \{B_{dx} B_j^l : 1 \leq j \leq l\} \), \( y \in Y \), and \( U \) be a neighborhood of \( y \) in \( Y \). We may assume that \( y \in X_0 \). Take \( B_{l+1} \in \mathcal{B}_l \) and \( k \in \mathbb{N} \) such that \( y \in B_{l+1}' = (B_{l+1} \times [0, 1/k)) \cap X \) and \( B_{l+1}' \cap Y \subset U \). Then we have

\[
B_{dx}(B_{l+1}' \cap Y) \subset \bigcap \{B_{dx} B_j^l : 1 \leq j \leq l + 1\}.
\]

By the induction hypothesis, we have

\[
\text{cmp} B_{dx}(B_{l+1}' \cap Y) \leq \text{cmp} \bigcap \{B_{dx} B_j^l : 1 \leq j \leq l + 1\} \leq m - (l + 1) = m - l - 1.
\]

Hence we have \( \text{cmp} Y \leq m - l \).

This completes the proof of the Claim.

By the Claim, in particular, \( \text{cmp} B_{dx}((B \times [0, 1/k)) \cap X) \leq m - 1 \) for every \( B \in \mathcal{B}_l \) and every \( k \in \mathbb{N} \). Since \( \{(B \times [0, 1/k)) \cap X : B \in \mathcal{B}_0 \cup \mathcal{B}_1 \text{ and } k \in \mathbb{N}\} \) is an outer base for \( X_0 \) in \( X \), we have \( \text{cmp} X \leq m \).

ADDED IN PROOF. By using the same techniques in §3, the author constructed a separable metrizable space \( X \) for which \( \text{cmp} X \neq \text{def} X \) (see [2]).

REFERENCES


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