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FAITHFUL ABELIAN GROUPS OF INFINITE RANK

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ABSTRACT. Let B be a subgroup of an abelian group G such that G/B is isomorphic to a direct sum of copies of an abelian group A. For B to be a direct summand of G, it is necessary that G be generated by B and all homomorphic images of A in G. However, if the functor $\mathrm{Hom}(A, -)$ preserves direct sums of copies of A, then this condition is sufficient too if and only if $M \otimes_{E(A)} A$ is nonzero for all nonzero right $E(A)$-modules M. Several examples and related results are given.

1. Introduction. There are only very few criteria for the splitting of exact sequences of torsion-free abelian groups. The most widely used of these was given by Baer in 1937 [**F**, Proposition 86.5]:

If C is a pure subgroup of a torsion-free abelian group G, such that G/C is homogeneous completely decomposable of type τ, and all elements of $G\backslash C$ are of type τ, then C is a direct summand of G.

Because of its numerous applications, many attempts have been made to extend the last result to situations in which G/C is not completely decomposable. Arnold and Lady succeeded in 1975 in the case that G is torsion-free of finite rank.

Before we can state their result, we introduce some additional notation: Suppose that A and G are abelian groups. The *A-socle* of G, denoted by $S_A(G)$, is the subgroup of G generated by $\{\phi(A) \mid \phi \in \mathrm{Hom}(A,G)\}$. The abelian group G is *A-projective* if it is isomorphic to a direct summand of $\bigoplus_I A$ for some index-set I. If I can be chosen to be finite, then G has *finite A-rank*. Finally, A is *faithful* if $IA \neq A$ for all proper right ideals I of the endomorphism ring, $E(A)$, of A.

Arnold and Lady showed in [**AL**, Theorem 2.1]: A torsion-free abelian group A of finite rank is faithful iff every exact sequence $0 \to B \xrightarrow{\alpha} G \xrightarrow{\beta} P \to 0$ of torsion-free abelian groups of finite rank with $G = \langle \alpha(B), S_A(G)\rangle$ and P A-projective is split-exact, where $\langle \alpha(B), S_A(G)\rangle$ denotes the subgroup of G which is generated by $\alpha(B)$ and $S_A(G)$.

Unfortunately, Arnold and Lady's proof uses induction on the rank of P to show that faithfulness is sufficient for the splitting of the sequences under consideration. This method breaks down if P has infinite A-rank. Nevertheless, Baer's classical theorem follows from Arnold and Lady's result in the finite rank case.

The goal of this paper is to discuss the class of abelian groups A such that every exact sequence $0 \to B \xrightarrow{\alpha} G \xrightarrow{\beta} P \to 0$ with P A-projective and $G = \langle S_A(G), \alpha(B)\rangle$

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splits. In order to remove the restrictions on the rank of A, we consider self-small abelian groups, i.e. groups A such that, for every index-set I and every $\phi \in \mathrm{Hom}(A, \bigoplus_I A)$, there is a finite subset I' of I with $\phi(A) \subseteq \bigoplus_{I'} A$. Obviously, a torsion-free group of finite rank is self-small, but the class of self-small groups contains, for instance, all abelian groups A with $E(A)$ countable [**AM**, Corollary 1.4]. Finally, A is *fully faithful* if $M \otimes_{E(A)} A \neq 0$ for all nonzero right $E(A)$-modules. M. The main result of this paper is

THEOREM 2.1. *The following are equivalent for a self-small abelian group A:*
(a) *A is fully faithful.*

(b) *Every exact sequence $0 \to B \xrightarrow{\alpha} G \xrightarrow{\beta} P \to 0$ with P A-projective and $G = \langle \alpha(B), S_A(G) \rangle$ splits.*

§3 will provide various examples that illustrate the previous results. In particular, A is self-small and faithful if $E(A)$ is a commutative or right principal subring of a finite dimensional $\mathbf{Q}$-algebra, or A is an E-ring.

2. Fully faithful abelian groups. Since we frequently use the basic properties of self-small abelian groups which have been discussed in [**AM**], we shall begin with a short summary of these. For reasons of simplicity, we denote the functor $\mathrm{Hom}(A, -)$ by H_A and the functor $- \otimes_{E(A)} A$ by T_A, where A is an abelian group. If G is an abelian group, and M is a right $E(A)$-module, then there is a map $\theta_G \colon T_A H_A(G) \to G$ defined by $\theta_G(\phi \otimes a) = \phi(a)$ and a map $\phi_M \colon M \to H_A T_A(M)$ defined by $[\phi_M(m)](a) = m \otimes a$ for all $\phi \in H_A(G)$, $m \in M$, and $a \in A$. In the case that A is self-small, θ_G is an isomorphism for all A-projective groups, while ϕ_M is an isomorphism for all projective right $E(A)$-modules M.

THEOREM 2.1. *The following are equivalent for a self-small abelian group A:*
(a) *A is fully faithful.*

(b) *Every exact sequence $0 \to B \xrightarrow{\alpha} G \xrightarrow{\beta} P \to 0$ of abelian groups with P A-projective and $G = \langle \alpha(B), S_A(G) \rangle$ splits.*

PROOF. (a) $\Rightarrow$ (b). Consider an exact sequence

$$0 \to B \xrightarrow{\alpha} G \xrightarrow{\beta} P \to 0 \tag{E1}$$

of abelian groups with P A-projective and $G = \langle \alpha(B), S_A(G) \rangle$. It induces the exact sequence

$$0 \to H_A(B) \to H_A(G) \to M \to 0 \tag{E2}$$

of right $E(A)$-modules where $M = \mathrm{im}\, H_A(\beta)$ is a submodule of the projective right $E(A)$-module $H_A(P)$.

To insure the splitting of (E2), it suffices to show $M = H_A(P)$. For this, define a map $\theta \colon T_A(M) \to P$ by $\theta(\phi \otimes a) = \phi(a)$ for all $\phi \in M$ and $a \in A$. An application of T_A to (E2) yields the top row of the commutative diagram

$$\begin{tikzcd}
T_A H_A(G) \arrow[r, "T_A H_A(\beta)"] \arrow[d, "\theta_G"] & T_A(M) \arrow[r] \arrow[d, "\theta"] & 0 \\
G \arrow[r, "\beta"] & P \arrow[r] & 0.
\end{tikzcd}$$

Since $S_A(G) = \operatorname{im}\theta_G$ and $G = \langle \alpha(B), S_A(G)\rangle$, the map $\beta\theta_G = \theta T_A H_A(\beta)$ is onto. Therefore, θ is onto. Furthermore, the inclusion map $i\colon M \to H_A(P)$ induces the commutative diagram

$$\begin{tikzcd} T_A(M) \arrow[r, "T_A(i)"] \arrow[d, equal] & T_A H_A(P) \arrow[r] \arrow[d, "\wr", "\theta_P"'] & T_A(H_A(P)/M) \arrow[r] & 0 \\ T_A(M) \arrow[r, "\theta"] & P \arrow[r] & 0. & \end{tikzcd}$$

The map θ_P is an isomorphism since A is self-small. Consequently, $T_A(i)$ is onto. Therefore, $T_A(H_A(P)/M) = 0$. By (a), $M = H_A(P)$.

Let $\phi\colon H_A(P) \to H_A(G)$ be a splitting map for $H_A(\beta)$. An application of the functor T_A yields the commutative diagram

$$\begin{tikzcd} T_A H_A(G) \arrow[d, "\theta_G"] & T_A H_A(P) \arrow[l, shift left, "T_A(\phi)"'] \arrow[d, "\wr", "\theta_P"'] \\ G \arrow[r, "\beta"] & P \end{tikzcd}$$

If $\nu = \theta_G T_A(\phi)\theta_P^{-1}$ then $\beta\nu = \mathrm{id}_P$, and (E1) splits.

(b) $\Rightarrow$ (a). Let M be a right $E(A)$-module with $T_A(M) = 0$. Choose a free resolution $0 \to U \xrightarrow{\alpha} F \xrightarrow{\beta} M \to 0$ where F is a free right $E(A)$-module. It induces the exact sequence $T_A(U) \xrightarrow{T_A(\alpha)} T_A(F) \to 0$ (E2). The $E(A)$-module U admits an exact sequence $P \xrightarrow{\delta} U \to 0$ with P projective. The exactness of the sequence $T_A(P) \xrightarrow{T_A(\delta)} T_A(U) \to 0$ yields an epimorphism $T_A(\alpha\delta)\colon T_A(P) \to T_A(F)$ which splits by (b). Consequently, the top row of the commutative diagram

$$\begin{tikzcd} H_A T_A(P) \arrow[r, "H_A T_A(\alpha\delta)"] & H_A T_A(F) \arrow[r] & 0 \\ P \arrow[u, "\wr", "\phi_P"'] \arrow[r, "\alpha\delta"] & F \arrow[u, "\phi_F", "\wr"'] & \end{tikzcd}$$

is exact. Hence, $\alpha\delta$ is onto, and the same holds for α. But this is only possible if $M = 0$.

A slight modification of the proof of the last result yields

COROLLARY 2.2. *The following are equivalent for a self-small abelian group* A:

(a) A *is faithful.*

(b) $T_A(M) \neq 0$ *for all finitely generated nonzero right* $E(A)$*-modules* M.

(c) *Every exact sequence* $0 \to B \xrightarrow{\alpha} G \xrightarrow{\beta} P \to 0$ *of abelian groups with* $G = \langle \alpha(B), S_A(G) \to$ *and* P A*-projective of finite* A*-rank splits.*

PROOF. (c) $\Rightarrow$ (b) is shown exactly as in the proof of Theorem 2.1 once we have observed that F can be chosen to be finitely generated.

(b) $\Rightarrow$ (a) is obvious since being faithful is equivalent to satisfying $T_A(M) \neq 0$ for all cyclic right $E(A)$-modules M.

(a) $\Rightarrow$ (c) is shown as in the proof of [**AL**, Theorem 2.1] by induction on the A-rank of P.

This last result contains [**AL**, Theorem 2.1] since every torsion-free group of finite rank is self-small. Moreover, if A is torsion-free of finite rank, then an A-projective group P has finite A-rank exactly if it has finite torsion-free rank.

Furthermore, if A is flat as an $E(A)$-module, then a right $E(A)$-module M satisfies $T_A(M) = 0$ iff $T_A(U) = 0$ for all finitely generated submodules U of M. Consequently, faithful and fully faithful are the same in this case:

COROLLARY 2.3. *The following are equivalent for a self-small abelian group A which is flat as an $E(A)$-module.*

(a) *A is faithful.*

(b) *Every exact sequence $0 \to B \xrightarrow{\alpha} G \xrightarrow{\beta} P \to 0$ with P A-projective and $G = \langle \alpha(B), S_A(G)\rangle$ splits.*

3. Examples. In [**A2**], we considered the property in Theorem 2.1(b) in conjunction with the following property: Every subgroup $B = S_A(B)$ of an A-projective group is A-projective. We showed that a self-small abelian group A has these two properties exactly if it is flat as an $E(A)$-module and has a right hereditary endomorphism ring. Consequently, we obtain

PROPOSITION 3.1. *A self-small abelian group A which is flat as an $E(A)$-module and has a right hereditary endomorphism ring is fully faithful.*

The conditions of Proportion 3.1 are satisfied for instance if A is a torsion-free reduced abelian group with a two-sided Noetherian hereditary endomorphism ring [**A1**, Proposition 3.2].

However, there are examples of faithful abelian groups which do not have hereditary endomorphism rings:

A ring R is an *E-ring* if every **Z**-endomorphisms of R^+ is multiplication by some element of R. E-rings were introduced by Schultz in [**S**] and Beaumont and Pierce in [**BP**].

Let Π denote the set of primes of **Z**. The ring $E = \Pi_{p\in\Pi}\mathbf{Z}/p\mathbf{Z}$ is an E-ring. It is self-injective since $\mathbf{Z}/p\mathbf{Z}$ is self-injective, and the class of self-injective rings is closed under products. Osofsky showed in [**O**] that a self-injective hereditary ring is semisimple Artinian. Therefore, E is an example of a nonhereditary E-ring.

EXAMPLE 3.2. Every E-ring R is self-small, fully faithful, and flat as an R-module.

PROOF. Clearly, R^+ is isomorphic to R as an R-module. Thus, $M \otimes_R R \cong M$, and R^+ is faithful and flat as an R-module. Furthermore, if $\phi \in \mathrm{Hom}(R, \bigoplus_I R)$, then there are $r_i \in R$ for $i \in I$ with $\phi(r) = (r_i r)_{i\in I}$. Moreover, $\phi(1) = (r_i)_{i\in I} \in \bigoplus_I R$ and hence $r_i = 0$ for almost all $i \in I$. Thus R is self-small.

Another class of examples is given by the next two results.

PROPOSITION 3.3. *A torsion-free abelian group A such that $E(A)$ is a right principal ideal ring, whose additive group has finite rank, is self-small and faithful.*

PROOF. Since $E(A)$ is countable, it remains to show that A is faithful. Let $I = \phi E(A)$ be a right ideal with $IA = A$. Consider the exact sequence $0 \to U \xrightarrow{\nu} A \xrightarrow{\phi} A \to 0$. An application of the functor $\mathrm{Hom}(-, A)$ yields the exact sequence

$$0 \to E(A) \xrightarrow{\mathrm{Hom}(\phi,A)} E(A) \xrightarrow{\mathrm{Hom}(\nu,A)} M \to 0$$

of left $E(A)$-modules where $M = \operatorname{im}(\operatorname{Hom}(\nu, A))$ is a submodule of $\operatorname{Hom}(U, A)$ which is torsion-free as an abelian group. Because of $r_0(E(A)) < \infty$, we have $M = 0$. Consequently, $\operatorname{Hom}(\phi, A)$ is an isomorphism. An application of the functor $\operatorname{Hom}_{E(A)}(-, A)$ yields the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_{E(A)}(E(A), A) \arrow[rr, "\operatorname{Hom}_{E(A)}(\operatorname{Hom}(\phi,A),A)"] & & \operatorname{Hom}_{E(A)}(E(A), A) \\ A \arrow[u, "\psi_A"', "\wr"] \arrow[rr, "\phi"] & & A \arrow[u, "\psi_A"', "\wr"] \end{tikzcd}$$

where the map ψ_A, which is defined by $[\psi_A(a)](\alpha) = \alpha(a)$ for all $\alpha \in E(A)$ and $a \in A$, is an isomorphism. Thus, ϕ is an isomorphism, and $I = E(A)$.

THEOREM 3.4. *A torsion-free reduced abelian group A such that $\mathbf{Q} \otimes_{\mathbf{Z}} E(A)$ is a commutative finite dimension $\mathbf{Q}$-algebra is self-small and faithful.*

PROOF. As in Proposition 3.3, it suffices to show that A is faithful. Let I be a right ideal of $E(A)$ with $A = IA$. If $E(A)/I$ is torsion, then there is a nonzero integer m with $mE(A) \subseteq I$. Consider the descending chain

$$I/mE(A) \supseteq (I/m^2E(A)) \supseteq \cdots \supseteq (I/m^nE(A)) \supseteq \cdots$$

of right ideals of the finite ring $E(A)/mE(A)$. There is some $n < \omega$ with

$$(I/mE(A))^n = (I/mE(A))^{n+1}.$$

By [**Ar1**, Lemma 5.8], there exists some $y \in I$ such that

$$(1 + y + mE(A))(I/mE(A))^n = 0.$$

In particular,

$$(1 + y + mE(A))(\langle I^n, mE(A)\rangle/mE(A)) = 0.$$

This yields $(1 + y)I^n \subseteq mE(A)$. Consequently, $mA \supseteq (1 + y)I^nA = (1 + y)(A)$ because of $IA = A$. For $a \in A$, choose $a' \in A$ with $(1 + y)(a) = ma'$ and define $\phi(a) = ma'$. Then, $1 + y = m\phi \in mE(A) \subseteq I$. Hence, $1 \in I$ and $I = E(A)$.

If $E(A)/I$ is not torsion, then $\mathbf{Q} \otimes_{\mathbf{Z}} I$ is a proper ideal of $\mathbf{Q} \otimes_{\mathbf{Z}} E(A)$. If we can show that every proper ideal of $\mathbf{Q} \otimes_{\mathbf{Z}} E(A)$ has a nonzero left annihilator, then there exists a nonzero integer m and $0 \neq \phi \in E(A)$ with $(1/m \otimes \phi)(\mathbf{Q} \otimes_{\mathbf{Z}} I) = 0$. Thus, $0 = \phi i A = \phi(A)$, a contradiction.

The ring $\mathbf{Q} \otimes_{\mathbf{Z}} E(A)$ is Artinian. Suppose that there exists a proper ideal J of $\mathbf{Q} \otimes_{\mathbf{Z}} E(A)$ which is minimal with respect to the property $\{\phi \in \mathbf{Q} \otimes_{\mathbf{Z}} E(A) | \phi J = 0\} = 0$. If $J^2 \subsetneq J$, then there is a nonzero element $\phi \in \mathbf{Q} \otimes_{\mathbf{Z}} E(A)$ with $\phi J^2 = 0$. Because of $\phi J \neq 0$, we obtain that J has a nonzero annihilator. The resulting contradiction shows $J^2 = J$. By [**Ar1**, Lemma 5.8], there is $y \in J$ with $(1+y)J = 0$ since J is finitely generated. Since J is proper, $1+y \neq 0$. Therefore, J has a nonzero annihilator. The resulting contradiction proves the lemma.

COROLLARY 3.5. ($V = L$). *Let κ be an uncountable cardinal number. There exists 2^κ many nonisomorphic self-small, fully faithful abelian groups of cardinality κ.*

PROOF. Let R be a torsion-free reduced ring as in Proposition 3.3 or Theorem 3.4. By [**DG**, Theorem 3.3], there are 2^κ many nonisomorphic abelian groups of cardinality κ with $E(A) \cong R$ such that every finite subset of A is contained in a free $E(A)$-submodule of A. The R-module A is flat because it is the direct limit of its finitely generated free submodules. Combining Proposition 3.3 and Theorem 3.4 with Corollary 2.3 and Theorem 2.1 yields that each such A is fully faithful.

Arnold has investigated torsion-free groups which are flat as an $E(A)$-modules [**Ar**, **AL**]. Lately, Goeters and Reid discovered an interesting characterization of abelian groups which are flat over their endomorphism ring and have finite rank [**GR**].

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