ALGEBRAIC MONOIDS WHOSE NONUNITS ARE PRODUCTS OF IDEMPOTENTS

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ABSTRACT. Let M be a connected regular linear algebraic monoid with zero and group of units G. Suppose G is nearly simple, i.e. the center of G is one dimensional and the derived group G' is a simple algebraic group. Then it is shown that S = M\G is an idempotent generated semigroup. If M has a unique nonzero minimal ideal, the converse is also proved. It follows that if G₀ is any simple algebraic group defined over an algebraically closed field K and if Φ: G₀ → GL(n,K) is any representation of G₀, then the nonunits of the monoid M(Φ) = K[a(G₀)] form an idempotent generated semigroup.

It has been shown by J. Erdos [3] (see also [2]) that any nonsingular matrix over a field is a product of idempotent matrices. Let K be an algebraically closed field. Our interest is in connected linear algebraic monoids with zero. This means by definition that the underlying set is an irreducible affine variety and that the product map is a morphism (i.e. a polynomial map). Let G denote the group of units of M, S = M\G. We are interested in knowing when S is idempotent generated. We will only consider the situation when G is a reductive group. This means [1] that the unipotent radical of G is trivial. Then by [8, 10], M is unit regular, i.e. M = E(M)G where E = E(M) = {e ∈ M|e² = e}. If X ⊆ M, let E(X) = X ∩ E(M) and ⟨X⟩ the semigroup generated by X. Let J,R,L,H denote the usual Green’s relations on M (see [5]). If a,b ∈ M, then a R b means MaM = MbM, a L b means aM = bM, a H b means Ma = Mb, H = R ∩ L. The following result was proved by the author [9, Theorem 2.7].

THEOREM 1 [9]. Suppose S = M\G has a maximum J-class (i.e. S is an irreducible variety). Then S is an idempotent generated semigroup.

Let R,G' denote the radical, derived group of G, respectively. Then [1, §1.8], R is contained in the center of G,G' is semisimple and G = RG'. Suppose dim R > 1. Then there exists e ∈ E(R), e ≠ 0,1. There exists g ∈ G such that eg ≠ e. If eg ∈ ⟨E(M)⟩, then eg = e₁...eₖ for some eᵢ ∈ E(M), i = 1,...,k. Then since e is a central idempotent, eg = e₁'...eₖ', where eᵢ' = eeᵢ ∈ E(M) and eᵢ' ≤ e. This is a contradiction since M is a matrix semigroup. Thus S = M\G is not idempotent generated. We will say that G is nearly simple if dim R = 1 and G' is a simple algebraic group, i.e. G' has no nontrivial closed connected normal subgroups.

THEOREM 2. Suppose G is nearly simple. Then S = M\G is an idempotent generated semigroup.

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PROOF. Let $R$ denote the radical of $G$, $G'$ the derived group of $G$. Since $\dim R = 1$, we see that $E(R) = \{0, 1\}$. Also, in any faithful linear representation of $M$, every element of $G'$ has determinant 1. It follows that $G'$ is closed in $M$. Let $e \in E = E(M)$, $e \neq 0, 1$. Let $G_e$ denote the connected component of the group $\{g \in G | ge = eg = e\}$. Then by [5], $e \in G_e$. Let $G_1$ denote the subgroup of $G$ generated by $x^{-1}Gex$ ($x \in G$). Then $G_1$ is a closed connected normal subgroup of $G$, $e \in G_1$. So $G_1 \subseteq G'$. Since $G$ is nearly simple, we see that $G_1 = G$. Let $E_0 = \langle E(J) \rangle$.

Let $H = \{g \in G | eg \in \langle E_0 \rangle\}$. Let $f \in E_0$, $g \in H$. By [6, Lemma 1.12], there exist $e_1, f_1 \in E_0$ such that $e \mathcal{L} e_1 \mathcal{R} f_1 \mathcal{L} f$. Then $e_1g = e_1(eg) \in \langle E_0 \rangle$. So $ge_1 = g(e_1g)g^{-1} \in \langle E_0 \rangle$. Hence $gf_1 = (ge_1)f_1 \in \langle E_0 \rangle$. So $f_1g = g^{-1}(gf_1)g \in \langle E_0 \rangle$. Thus $fg = f(f_1g) \in \langle E_0 \rangle$. It follows that $E_0H \subseteq \langle E_0 \rangle$. In particular $x^{-1}Hx = H$ for all $x \in G$. Now let $a, b \in H$. Then $ea, eb \in \langle E_0 \rangle$. So $be = (be)(ea) \in \langle E_0 \rangle$. Hence $eab = b^{-1}(bea)b \in \langle E_0 \rangle$. Thus $S \subseteq \langle E \rangle$.

COROLLARY 1. Let $G_0$ be a simple algebraic group, $\Phi: G_0 \to GL(n, K)$ any representation. Let $M(\Phi)$ denote the Zariski closure of $K\Phi(G_0)$ in $\mathcal{M}_n(K)$. Then the nonunits of $M(\Phi)$ form an idempotent generated semigroup.

We now consider the situation dual to that in Theorem 1, namely when $M \setminus \{0\}$ has a minimum $\mathcal{J}$-class. Equivalently, the intersection of all nonzero ideals of $M$ is again nonzero. Renner [11, Corollary 8.3.3] has characterized these monoids as being exactly those having an idempotent separating (equivalently finite in the algebraic geometry sense) irreducible, linear representation. Moreover, in such a case the radical $R$ of $G$ is one dimensional [11, Lemma 8.3.2].

THEOREM 3. Suppose $M$ has a unique nonzero minimal ideal. Then $S = M \setminus G$ is an idempotent generated semigroup if and only if $G$ is nearly simple.

PROOF. Let $R$ denote the one dimensional radical of $G$ and $J_0$ the nonzero minimum $\mathcal{J}$-class of $M$. So $J_0 \cup \{0\}$ is the minimum nonzero ideal of $M$. If $G$ is nearly simple, then by Theorem 2, $S$ is idempotent generated. Conversely suppose that $S \subseteq \langle E \rangle$ where $E = E(M)$. Suppose $G$ is not nearly simple. Then $G = G_1G_2$ where $G_1, G_2$ are proper closed connected normal subgroups of $G$ containing $R$, and such that $g_1g_2 = g_2g_1$ for all $g_i \in G_i, i = 1, 2$. Let $M_i$ denote the closure of $G_i$ in $M$, $i = 1, 2$. Since $R \neq G_1$, there exists $e \in E(M_1), e \neq 0, 1$. By [5], $J = GeG$ is the $\mathcal{J}$-class of $e$ in $M$. By [5], any element of $E(J)$ is conjugate to $e$. Since $G_2$ centralizes $M_1$ and $G = G_1G_2$, we see that $E(J) \subseteq M_1$. Let $g \in G$, $a = eg$. Then $a = e_1 \cdots e_m$ for some $e_1, \ldots, e_m \in E$. Let

$$e'_i = e_i \cdots e_m g^{-1} e e_1 \cdots e_i \in E(J), \quad i = 1, \ldots, m.$$ 

Then with $e_0 = e$, we have for $i = 1, \ldots, m$,

$$e_0 \cdots e_{i-1} e'_i = e_0 \cdots e_m g^{-1} e e_1 \cdots e_i = e g^{-1} e e_1 \cdots e_i = e e_1 \cdots e_i.$$ 

So $e e'_1 \cdots e'_m = e e_1 \cdots e_m = a$. Thus $a \in \langle E(J) \rangle \subseteq M_1$. Hence $ge = gag^{-1} \in \langle E(J) \rangle \subseteq M_1$. It follows that $J = GeG \subseteq M_1$. Hence $MeM = J \subseteq M_1$. Thus $J_0 \subseteq M_1$. So $J_0$ is centralized by $G_2$. Similarly $J_0 \subseteq M_2$ and $J_0$ is centralized by
Thus $J_0$ is centralized by $G$. So if $f \in E(J_0)$, then by [7, Theorem 2.3], $f \in \overline{R}$. This contradicts the fact that $\dim R = 1$, proving the theorem.

The difference between Theorems 1 and 3 is illustrated by the following elementary examples.

**Example 1.** Let $M = \{A \otimes B | A, B \in \mathcal{M}_2(K), \det A = \det B\}$, $G$ the group of units of $M$. Then $S = M \backslash G$ has a maximum $\mathcal{J}$-class and by Theorem 1, $S$ is idempotent generated. Note that $G$ is not nearly simple.

**Example 2.** Let $M = \{A \otimes B | A, B \in \mathcal{M}_2(K)\}$, $G$ the group of units of $M$, $S = M \backslash G$. Then $M$ has a unique nonzero minimal ideal. Clearly $G$ is not nearly simple and hence by Theorem 3, $S$ is not idempotent generated.

**References**