THE SPACES $H^p(B_n), 0 < p < 1$, AND $B_{pq}(B_n), 0 < p < q < 1$, ARE NOT LOCALLY CONVEX

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ABSTRACT. In this note, we use the Ryll-Wojtaszczyk polynomials to prove that the spaces $H^p(B_n), 0 < p < 1$, and $B_{pq}(B_n), 0 < p < q < 1$, fail to be locally convex.

Let $B_n$ denote the unit ball of complex vector space $\mathbb{C}^n$, $\partial B_n$ its boundary and $\sigma_n$ the positive rotation-invariant measure on $\partial B_n$ with $\sigma_n(\partial B_n) = 1$. By $H(B_n)$ we denote the class of all functions holomorphic in $B_n$.

The Hardy space $H^p, 0 < p < \infty$, is defined on $B_n$ by

$$H^p(B_n) = \{f : f \in H(B_n), \|f\|_p < \infty\},$$

where

$$\|f\|_p = \sup_{0<r<1} \ M_p(r, f), \quad M_p(r, f) = \left\{ \int_{\partial B_n} |f(\zeta)|^p \, d\sigma_n(\zeta) \right\}^{1/p}.$$

The space $B_{pq}, 0 < p < q < \infty$, is defined on $B_n$ by

$$B_{pq}(B_n) = \{f : f \in H(B_n), \|f\|_{B_{pq}} < \infty\},$$

where

$$\|f\|_{B_{pq}} = \left\{ \int_0^1 (1-r)^{nq(1/p-1/q)} \ M_q^q(r, f) \, dr \right\}^{1/q}.$$

For $p \geq 1$, $H^p(B_n)$ is a Banach space with norm (1) and for $q \geq 1$, $B_{pq}(B_n)$ is a Banach space with norm (2). For $0 < p < 1$ and $0 < p < q < 1$, we introduce the metrics

$$d(f, g) = \|f - g\|^p_p \quad \text{for } H^p(B_n)$$

and

$$d(f, g) = \|f - g\|_{B_{pq}}^q \quad \text{for } B_{pq}(B_n).$$

These metrics turn $H^p(B_n)$ and $B_{pq}(B_n)$ respectively into $F$-spaces (complete, metrizable linear topological spaces).

Livingston first noticed that $H^p(U)$ fails to be locally convex for $0 < p < 1$ and $U$ a unit disc, [2]. In this note we will prove that $H^p(B_n), 0 < p < 1$, and $B_{pq}(B_n), 0 < p < q < 1$, are not locally convex.

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Let $f \in H(B_n)$ with the homogeneous expansion $f(z) = \sum_{k=0}^\infty F_k(z)$ and $\beta > 0$; the $\beta$th fractional derivative and fractional integral of $f$ are defined respectively by

$$f^{[\beta]}(z) = \sum_{k=0}^\infty \frac{\Gamma(k + \beta + 1)}{\Gamma(k + 1)} F_k(z)$$

and

$$f_{[\beta]}(z) = \sum_{k=0}^\infty \frac{\Gamma(k + 1)}{\Gamma(k + \beta + 1)} F_k(z).$$

In the above, $\Gamma$ denotes the Gamma function. It is known that $f^{[\beta]}$ and $f_{[\beta]}$ are holomorphic on $B_n$ [3].

In 1983 Ryll and Wojtaszczyk [5] proved the following theorem:

For every $n \geq 1$, there is a sequence $p_1, p_2, \ldots$ with $p_k \in P(k, n)$, such that

$$\|p_k\|_{\infty} = 1, \quad \|p_k\|_2 \geq \sqrt{\pi/2n}, \quad k = 1, 2, \ldots,$$

where $P(k, n)$ is the vector space of homogeneous polynomials of degree $k$ in the complex variables $z_1, \ldots, z_n$ regarded as functions on $\partial B_n$ and

$$\|p_k\|_2 = \left\{ \int_{\partial B_n} |p_k(z)|^2 \, d\sigma_n(z) \right\}^{1/2}.$$

\{p_k\}, $k = 1, 2, \ldots$, are called the Ryll-Wojtaszczyk polynomials, for which we have

**Lemma 1.** Let $\{n_k\}$ be a lacunary sequence of positive integers and $\{p_{n_k}\}$ the Ryll-Wojtaszczyk polynomials. If $f(z) = \sum_{k=1}^\infty a_k p_{n_k}(z)$ with $\sum_{k=1}^\infty |a_k|^2 < \infty$, then for any $p$, $0 < p < 1$, there is a constant $A$ independent of $f$, such that

$$\|f\|_p \leq A \|f\|_p.$$

**Proof.** Since $|p_{n_k}(z)| \leq 1$, $z \in \partial B_n$, $\|p_{n_k}\|_1 \geq \|p_{n_k}\|_2^2 \geq \pi/4^n$. Proposition 1.6 of [5] gives

$$\left\{ \sum_{k=1}^\infty |a_k|^2 \right\}^{1/2} \leq A \|f\|_1.$$

Here, and later, $A$ denotes a finite constant, not necessarily the same at each occurrence, independent of $f$. It is easy to see

$$\|f\|_2^2 \leq \sum_{k=1}^\infty |a_k|^2 \|p_{n_k}\|_2^2 \leq \sum_{k=1}^\infty |a_k|^2.$$

Combining (3) and (4) we obtain

$$\|f\|_2 \leq A \|f\|_1.$$

On the other hand, for any $p$, $0 < p < 1$, Hölder’s inequality gives

$$\|f\|_1 \leq \|f\|_p^{(2-p)/2} \|f\|_2^{(2-p)/(2-p)}.$$

From (5) and (6), we have

$$\|f\|_2 \leq A \|f\|_p.$$

Now (3) and (7) give the desired inequality.
A function $f$, holomorphic in $B_n$ and continuous on $\overline{B_n}$, belongs to Lipschitz class $\Lambda_\alpha$, $0 < \alpha \leq 1$, if it satisfies the Lipschitz condition

$$|f(\zeta) - f(\eta)| \leq A|\zeta - \eta|^{\alpha}, \quad |\zeta - \eta| \to 0,$$

$\zeta, \eta \in \partial B_n$, $A$ independent of $\zeta, \eta$.

**Lemma 2.** Let $f(z) = \sum_{k=0}^{\infty} F_k(z)$ be the homogeneous expansion of $f$ on $B_n$. If $f \in \Lambda_\alpha$, $0 < \alpha \leq 1$, then $\sum_{k=1}^{\infty} k^s \|F_k\|^2 < \infty$ for any $s < 2\alpha$.

**Proof.** Since $f \in \Lambda_\alpha$, by Theorem 6 of [3],

$$|f^{(1)}(r\zeta)| \leq A(1-r)^{\alpha-1}, \quad 0 < r < 1, \quad \zeta \in \partial B_n,$$

where $A$ is independent of $r$ and $\zeta$. Inequality (8) and

$$\int_{\partial B_n} \left| f^{(1)}(r\zeta) \right|^2 d\sigma_n(\zeta) = \sum_{k=0}^{\infty} (k + 1)^2 r^{2k} \|F_k\|^2$$

give $r^{2N} \sum_{k=1}^{N} k^2 \|F_k\|^2 \leq A(1-r)^{2(\alpha-1)}$. Let $r = 1 - N^{-1}$; we have

$$S_N = \sum_{k=1}^{N} k^2 \|F_k\|^2 \leq AN^{2(1-\alpha)}.$$ 

Hence

$$\sum_{k=1}^{N} k^s \|F_k\|^2 = \sum_{k=1}^{N} k^2 \|F_k\|^2 k^{s-2} = \sum_{k=1}^{N-1} S_k [k^{s-2} - (k+1)^{s-2}] + S_N N^{s-2} \leq A \sum_{k=1}^{N-1} [k^{s-2\alpha} - (k+1)^{s-2\alpha}] + O(1) = O(1)$$

for $s < 2\alpha$. The lemma is proved.

**Theorem 1.** $H^p(B_n), 0 < p < 1,$ is not locally convex.

**Proof.** Let

$$M = \left\{ f(z) = \sum_{k=1}^{\infty} a_k p_{n_k}(z) : \sum_{k=1}^{\infty} |a_k|^2 < \infty \right\},$$

where $\{n_k\}$ is a lacunary sequence of natural numbers and $\{p_{n_k}\}$ are the Ryll-Wojtaszczyk polynomials. It is clear that $M$ is a subspace of $H^p(B_n), 0 < p < 1$. Take a sequence $\{c_k\}$ such that $\sum_{k=1}^{\infty} |c_k|^2 < \infty$ and for any $\varepsilon > 0$, $\sum_{k=1}^{\infty} k^\varepsilon |c_k|^2 = \infty$. Define a linear functional $\varphi$ on $M$ as follows:

$$\varphi(f) = \sum_{k=1}^{\infty} a_k c_k,$$

$$f(z) = \sum_{k=1}^{\infty} a_k p_{n_k}(z) \in M.$$

The Schwarz Lemma and Lemma 1 give

$$|\varphi(f)| \leq A \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \leq A\|f\|_p, \quad f \in M.$$
This proves that \( \varphi \) is a continuous linear functional on \( M \). If \( H^p(B_n), 0 < p < 1 \), is locally convex, then there exists a continuous linear functional \( \Phi \) on \( H^p(B_n) \), such that \( \Phi(f) = \varphi(f) \) on \( M \) [4, Theorem 3.6]. By Theorem 9 of [3], there is some \( g \in \Lambda_\alpha, 0 < \alpha \leq 1 \), such that

\[
\Phi(f) = \lim_{r \to 1} \int_{\partial B_n} f(r\zeta)\overline{g(\zeta)}\,d\sigma_n(\zeta)
\]

for all \( f \in H^p(B_n) \). Let \( g(z) = \sum_{k=0}^{\infty} F_k(z) \); Lemma 2 gives

\[
\sum_{k=1}^{\infty} k^s\|F_k\|^2 < \infty, \quad s < 2\alpha.
\]

Take \( f = p_{n_k} \) in (9); since \( \Phi(f) = \varphi(f) = c_k \), we have \( |c_k| \leq \|F_{n_k}\|_2, k = 1, 2, \ldots \). Now (10) gives

\[
\sum_{k=1}^{\infty} k^s|c_k|^2 \leq \sum_{k=1}^{\infty} n_k^s\|F_{n_k}\|^2 < \infty, \quad s < 2\alpha.
\]

This contradiction proves that \( H^p(B_n) \) fails to be locally convex.

**THEOREM 2.** \( B_{pq}(B_n), 0 < p < q < 1 \), is not locally convex.

Before proving the theorem, we first prove the following lemma.

**LEMMA 3.** Suppose \( 0 < p < q < 1 \) and \( \beta = n(1/p - 1/q) \). If \( f^{[\beta]} \in B_{pq}(B_n) \), then \( f \in H^q(B_n) \) and there exists a constant \( A \) independent of \( f \), such that \( \|f\|_q \leq A\|f^{[\beta]}\|_{B_{pq}} \).

**PROOF.** A direct computation gives

\[
f(r\zeta) = \frac{1}{\Gamma(\beta)} \int_0^1 (1 - \rho)^{\beta-1} f^{[\beta]}(r\rho \zeta) \,d\rho
\]

for \( 0 < r < 1 \) and \( \beta > 0 \), and

\[
|f(r\zeta)| \leq \frac{1}{\Gamma(\beta)} \sum_{k=1}^{\infty} \int_{\rho_{k-1}}^{\rho_k} (1 - \rho)^{\beta-1}|f^{[\beta]}(r\rho \zeta)| \,d\rho,
\]

where \( \rho_k = 1 - 2^{-k} \). Let \( H_k(r\zeta) = \sup\{|f^{[\beta]}(r\rho \zeta)|: \rho_{k-1} < \rho < \rho_k\} \). Then

\[
|f(r\zeta)|^q \leq A \sum_{k=1}^{\infty} H_k^q(r\zeta)2^{-q\beta k}.
\]

Here we have used the condition \( 0 < q < 1 \). Now Theorem 3 of [1] and the monotonicity of the mean give

\[
M_q^q(r, f) \leq A \sum_{k=1}^{\infty} 2^{-q\beta k} \sup\{M_q^q(r\rho, f^{[\beta]}): \rho_{k-1} < \rho < \rho_k\}
\]

\[
\leq A \sum_{k=1}^{\infty} 2^{-q\beta k} M_q^q(r\rho_k, f^{[\beta]}) \leq A \int_0^1 (1 - \rho)^{\beta_q-1} M_q^q(\rho, f^{[\beta]}) \,d\rho.
\]

Taking \( \beta = n(1/p - 1/q) \) in the above inequality, the desired result follows.
Proof of Theorem 2. Let $\beta = n(1/p - 1/q)$ and $E = \{h: h \in H^p(B_n),
 h[\beta] \in M\}$. Taking $k = q$ in Theorem 4 of [3], we have

\[ \|f\|_{B_{pq}} \leq A\|f\|_p, \quad 0 < p < q < \infty. \]  (12)

This shows that $E$ is a subspace of $B_{pq}(B_n)$. It is clear that for every $h \in E$, there exists an $f \in M$ with $f[\beta] = h$, hence $h$ can be written as

\[ h(z) = \sum_{k=1}^{\infty} \frac{\Gamma(n_k + \beta + 1)}{\Gamma(n_k + 1)} a_k p_{n_k}(z). \]  (13)

Take a sequence $\{b_k\}$ such that $\sum_{k=1}^{\infty} n_k^2 |b_k|^2 < \infty$, and $\sum_{k=1}^{\infty} n_k^{2\beta + \epsilon} |b_k|^2 = \infty$ for any $\epsilon > 0$. Define a linear functional on $E$ as follows:

\[ \psi(h) = \sum_{k=1}^{\infty} \frac{\Gamma(n_k + \beta + 1)}{\Gamma(n_k + 1)} a_k b_k, \quad h \in E. \]  (14)

The series in (14) is convergent since $\Gamma(n_k + \beta + 1)/\Gamma(n_k + 1) \approx n_k^\beta$ and

\[ |\psi(h)| \leq A \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}, \quad h \in E. \]  (15)

Since $f[\beta] = h \in H^p(B_n) \subset B_{pq}(B_n)$, according to Lemma 3, there is a constant $A$ such that

\[ \|f\|_q \leq A\|f[\beta]\|_{B_{pq}} = A\|h\|_{B_{pq}}. \]  (16)

(15), (16) and Lemma 1 give $|\psi(h)| \leq A\|h\|_{B_{pq}}$ for $h \in E$. This proves that $\psi$ is a continuous linear functional on the subspace $E$ of $B_{pq}(B_n)$. If $B_{pq}(B_n)$ is locally convex, then there is a continuous linear functional $\Psi$ on $B_{pq}(B_n)$ with $\Psi(h) = \psi(h)$ on $E$. By (12), $\Psi$ is also a continuous linear functional on $H^p(B_n)$.

Suppose $p$ satisfies $n/(n + 1) < p < n/(n + l - 1)$, $l$ a natural number. By Theorem 9 of [3], there exists a $v$ with $v[l-1] \in \Lambda_\alpha$, $\alpha = n(1/p - 1) - l + 1$, such that

\[ \Psi(h) = \lim_{r \to 1} \int_{\partial B_n} h(rz) \overline{v(z)} \sigma_n(\xi) \]  (17)

for every $h \in H^p(B_n)$. Let $v(z) = \sum_{k=0}^{\infty} G_k(z)$; Lemma 2 tells us

\[ \sum_{k=1}^{\infty} k^{s+2(l-1)} \|G_k\|^2 < \infty, \quad s < 2\alpha. \]  (18)

Take $h = (\Gamma(n_k + \beta + 1)/\Gamma(n_k + 1)) p_{n_k}$ in (17); since

\[ \Psi(h) = \psi(h) = (\Gamma(n_k + \beta + 1)/\Gamma(n_k + 1)) b_k, \]

we have

\[ |b_k| \leq \|G_{n_k}\|_2, \quad k = 1, 2, \ldots. \]  (19)

Choose $\eta > 0$ such that $\eta + 2\beta < 2n(1/p - 1) - \eta$ and let $s = 2\alpha - \eta$. Then $s + 2(l - 1) = 2n(1/p - 1) - \eta > \eta + 2\beta$. Now (18) and (19) give

\[ \sum_{k=1}^{\infty} n_k^{\eta+2\beta} |b_k|^2 \leq \sum_{k=1}^{\infty} n_k^{s+2(l-1)} \|G_{n_k}\|^2_2 < \infty. \]

This is a contradiction. The theorem is proved.
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