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CHAIN-PRESERVING DIFFEOMORPHISMS AND CR EQUIVALENCE

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ABSTRACT. It is shown that a diffeomorphism that preserves chains between two nondegenerate CR manifolds is actually either a CR isomorphism or a conjugate CR isomorphism.

Recently, H. Jacobowitz [**4**] showed that any two sufficiently closed points on a strictly pseudo-convex CR manifold (of hypersurface type) can be connected by a smooth chain. In this regard, the chains do behave like geodesics in Riemannian geometry. The purpose of this note is to point out that in another regard the chains do not behave like geodesics in Riemannian geometry but behave like conformal circles in conformal geometry [**6**]. Namely, we prove that a (local) diffeomorphism that preserves chains between two nondegenerate CR manifolds (not necessarily of the same signature type a priori) is actually either a CR isomorphism or a conjugate CR isomorphism. (It follows that these two CR manifolds have to be of the same signature type.) The analogous statement is known to hold in either projective geometry or conformal geometry but not in Riemannian geometry. The significance of this type of results is to reduce the equivalence problem of the original geometry to that of geometry of special invariant curves defined by a system of ordinary differential equations. It should be mentioned that our result was first claimed by Professor T. Nagano even for certain class of geometries which generalize the CR structure. (See the remark in the end of §2.)

1. Let us start with a brief outline of CR geometry. Let M be an odd dimensional (smooth) manifold of dimension $2n+1$, $n \geq 1$. Let G be the group of matrices

$$\begin{pmatrix} u & 0 & 0 \\ v^{\alpha} & U^{\alpha}_{\beta} & 0 \\ v^{\bar{\alpha}} & 0 & U^{\bar{\alpha}}_{\bar{\beta}} \end{pmatrix}, \quad v^{\bar{\alpha}} = \overline{v^{\alpha}}, \quad U^{\bar{\alpha}}_{\bar{\beta}} = \overline{U^{\alpha}_{\beta}}, \qquad \det(U^{\alpha}_{\beta}) \neq 0,$$

where, as throughout this section, the small Greek indices run from 1 to n, u is real, and v^{α}, U^{α}_{β} are complex. Note that G can be considered as a subgroup of $GL(2n+1, \mathbf{R})$. A G-structure on M is a reduction of the group of its tangent bundle to G. Locally it is given by an admissible coframe θ, θ^{α}, $\theta^{\bar{\alpha}}$, where θ is real

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and θ^α's are complex. The G-structure is called integrable if $d\theta$, $d\theta^\alpha$ belong to the differential ideal generated by θ, θ^β. Since θ is real, the above condition implies

$$d\theta = i g_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} + \theta \wedge \phi \tag{1.1}$$

(the summation convention is used from now on) for some real one form ϕ and $g_{\alpha\bar\beta} = \overline{g_{\beta\bar\alpha}}$. A CR (Cauchy-Riemann) structure on M is an integrable G-structure. A manifold M with a CR structure is called a CR manifold. A diffeomorphism between two CR manifolds is a CR isomorphism if it is an isomorphism as G-structure. A CR structure is called nondegenerate if $\det(g_{\alpha\bar\beta}) \neq 0$. Given an admissible local coframe $(\theta, \theta^\alpha, \theta^{\bar\alpha})$ and ϕ satisfying (1.1), there associate uniquely determined connection forms ϕ_β^α, ϕ^α, ψ satisfying $dg_{\alpha\bar\beta} = \phi_\alpha^\tau g_{\tau\bar\beta} + g_{\alpha\bar\tau}\phi_{\bar\beta}^{\bar\tau} - g_{\alpha\bar\beta}\phi$ and some trace conditions. (See the original papers of E. Cartan [**2**] for $n = 1$ and S. Chern and J. Moser [**3**] for general n.) Two admissible local coframes and their associated connection forms are related according to the formula

$$\tilde\pi = dh \cdot h^{-1} + h\pi h^{-1}, \tag{1.2}$$

where

$$\pi = \begin{pmatrix} \pi_0^0 & \theta^\alpha & 2\theta \\ -i\phi_\alpha & \phi_\alpha^\beta + \delta_\alpha^\beta \pi_0^0 & 2i\theta_\alpha \\ -\frac{1}{4}\psi & \frac{1}{2}\phi^\alpha & -\overline{\pi_0^0} \end{pmatrix},$$

$\pi_0^0 = (-1/(n+2))(\phi_\alpha^\alpha + \phi)$ and $g_{\alpha\bar\beta}$ is used to raise or lower indices and assumed to be constant. For instance, $\theta_\alpha = g_{\alpha\bar\beta}\theta^{\bar\beta}$. Since we are always dealing with nondegenerate CR structures, we may just assume

$$(g_{\alpha\bar\beta}) = \begin{pmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{p} & \\ & \overbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}^{q} \end{pmatrix} \quad \text{(p positive eigenvalues and q negative eigenvalues)}, \tag{1.3}$$

$$h = \begin{pmatrix} t & 0 & 0 \\ t_\alpha & t_\alpha^\beta & 0 \\ \tau & \tau^\beta & \bar t^{-1} \end{pmatrix}$$

subject to

$$t_\alpha = -2it t_\alpha^\rho \tau^{\bar\sigma} g_{\rho\bar\sigma},$$
$$t\bar t^{-1} \det(t_\alpha^\beta) = 1,$$
$$t_\alpha^\rho \bar t_\beta^\sigma g_{\rho\bar\sigma} = g_{\alpha\bar\beta},$$
$$g_{\rho\bar\sigma}\tau^\rho\tau^{\bar\sigma} + (i/2)(\bar\tau\bar t^{-1} - \tau t^{-1}) = 0.$$

Now, a chain on M is a curve on which $\tilde\theta^\alpha = 0$, $\tilde\phi^\alpha = 0$ for some suitably chosen admissible local coframe $(\tilde\theta, \tilde\theta^\alpha, \overline{\tilde\theta^\alpha})$ and $\tilde\phi$ satisfying (1.1). If we fix an admissible local coframe $(\theta, \theta^\alpha, \overline{\theta^\alpha})$ and ϕ satisfying (1.1) with $g_{\alpha\bar\beta}$ as in (1.3), the equations of the chains can be obtained through (1.2) by taking h of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2i{}^t\bar\sigma_+ & I_p & 0 & 0 \\ 2i{}^t\bar\sigma_- & 0 & I_q & 0 \\ i(|\sigma_-|^2 - |\sigma_+|^2) & \sigma_+ & \sigma_- & 1 \end{pmatrix},$$

where $\sigma_+ = (\sigma^{\alpha_+})$ and $\sigma_- = (\sigma^{\alpha_-})$, $1 \le \alpha_+ \le p$, $p+1 \le \alpha_- \le p+q = n$, are row vectors, ${}^t\overline{\sigma_+}$ and ${}^t\overline{\sigma_-}$ are the corresponding conjugate column vectors and $|\sigma_-|^2$ ($|\sigma_+|^2$ resp.) denotes the square length of σ_- (σ_+ resp.). (See [**1**] for details in the case of strictly pseudoconvex M. And it is easy to generalize the argument on p. 165 of [**1**] to the nondegenerate case. Note that there are sign changes for (1.1) and σ^α in [**1**].) Setting $\tilde\theta^\alpha = 0$, $\tilde\phi^\alpha = 0$ in (1.2), we achieve the following equations of the chains:

$$\begin{aligned} \theta^{\alpha_+} &= 2\sigma^{\alpha_+}\theta, \quad \theta^{\alpha_-} = 2\sigma^{\alpha_-}\theta, \qquad 1 \le \alpha_+ \le p;\ p+1 \le \alpha_- \le p+q = n, \\ d\sigma^{\alpha_+} &= 4i\sigma^{\alpha_+}(|\sigma_+|^2 - |\sigma_-|^2)\theta - \sigma^{\alpha_+}\overline{\pi_0^0} \\ &\quad + (\phi_{\bar\alpha_+}^{\bar\beta_+} + \delta_{\alpha_+}^{\beta_+}\overline{\pi_0^0})\sigma^{\beta_+} - \phi_{\overline{\alpha_+}}^{\bar\beta_-}\sigma^{\beta_-} - \bar\phi_{\alpha_+}/2, \\ d\sigma^{\alpha_-} &= 4i\sigma^{\alpha_-}(|\sigma_+|^2 - |\sigma_-|^2)\theta - \sigma^{\alpha_-}\overline{\pi_0^0} \\ &\quad - \phi_{\overline{\alpha_-}}^{\bar\beta_+}\sigma^{\beta_+} + (\phi_{\bar\alpha_-}^{\bar\beta_-} + \delta_{\alpha_-}^{\beta_-}\overline{\pi_0^0})\sigma^{\beta_-} + \overline{\phi_{\alpha_-}}/2. \end{aligned} \tag{1.4}$$

where β_+ (β_- resp.) runs from 1 to p ($p+1$ to n, resp.). Given a nondegenerate CR structure with an admissible coframe $(\theta, \theta^\alpha, \overline{\theta^\alpha})$ locally. Let us define its conjugate CR structure by assigning the admissible coframe: $\tilde\theta = \theta$, $\tilde\theta^\alpha = \overline{\theta^\alpha}$, $\overline{\tilde\theta^\alpha} = \theta^\alpha$. It is easily seen that the conjugation of the connection forms associated to the coframe $(\theta, \theta^\alpha, \overline{\theta^\alpha}, \phi)$ satisfying (1.1) gives the corresponidng set of connection forms for $(\tilde\theta, \tilde\theta^\alpha, \overline{\tilde\theta^\alpha}, \tilde\phi = \phi)$. And taking conjugation of (1.1), we will not change the signature of $(g_{\alpha\bar\beta})$ as the conjugate complex structure is signaled by $-i$. So the conjugate CR structure has the same signature type as the original nondegenerate CR structure. A diffeomorphism between two CR manioflds M and $\tilde M$ is called a conjugate CR isomorphism between M with the given CR structure (its conjugate CR structure resp.) and $\tilde M$ with its conjugate CR structure (the given CR structure resp.) Now, we observe that the chains for the conjugate CR structure coincide with those for the original CR structure as taking conjugation of each equation in (1.4) shows.

The converse is our main result.

THEOREM. *A (local) diffeomorphism that preserves chains between two nondegenerate CR manifolds (not necessarily of the same signature type a priori) is actually either a CR isomorphism or a conjugate CR isomorphism.*

2. In this section, we are going to give a proof of our theorem. Since the connection forms can be expressed in the base $\theta, \theta^\alpha, \theta^{\bar\alpha}$, using the first two equations of (1.4), we may rewrite (1.4) in a simple form.

$$\begin{aligned} \theta^\alpha &= \sigma^\alpha\theta, \qquad 1 \le \alpha \le n, \\ d\sigma^\alpha &= i\sigma^\alpha(|\sigma_+|^2 - |\sigma_-|^2)\theta + B^\alpha\theta, \end{aligned} \tag{2.1}$$

where B^α is a polynomial of degree less than or equal to 2 in variables σ^α, $\overline{\sigma^\alpha}$. Note that $2\sigma^\alpha$ has been replaced by σ^α.

Consider another nondegenerate CR manifold $\widetilde{M}$. Let $\tilde{\theta}$, $\tilde{\theta}^\alpha$, $\overline{\tilde{\theta}^\alpha}$ be a local admissible coframe for $\widetilde{M}$ satisfying

$$d\tilde{\theta} = i\tilde{g}_{\alpha\bar{\beta}}\tilde{\theta}^\alpha \wedge \overline{\tilde{\theta}^\beta} + \tilde{\theta} \wedge \tilde{\phi}$$

for some real one form $\tilde{\phi}$ and

$$(\tilde{g}_{\alpha\bar{\beta}}) = \begin{pmatrix} \overbrace{1 \quad \cdot \quad \cdot \quad 1}^{\tilde{p}} & \\ & \overbrace{-1 \quad \cdot \quad \cdot \quad -1}^{\tilde{q}} \end{pmatrix}$$

($\tilde{p}$ positive eigenvalues and $\tilde{q}$ negative eigenvalues).

Note that $\tilde{p}$ ($\tilde{q}$ resp.) may not be equal to p (q resp.). The equations of the chains with respect to $(\tilde{\theta}, \tilde{\theta}^\alpha, \overline{\tilde{\theta}^\alpha})$ take the similar form as (2.1).

$$\begin{aligned} \tilde{\theta}^\alpha &= \tilde{\sigma}^\alpha \tilde{\theta}, \quad 1 \le \alpha \le n, \\ d\tilde{\sigma}^\alpha &= i\tilde{\sigma}^\alpha(|\tilde{\sigma}_+|^2 - |\tilde{\sigma}_-|^2)\tilde{\theta} + \tilde{B}^\alpha\tilde{\theta}. \end{aligned} \tag{2.2}$$

Now, suppose a local diffeomorphism f from M to $\tilde{M}$ preserves chains. Using the pullback $f^*\tilde{\theta}$, $f^*\tilde{\theta}^\alpha$, $f^*\overline{\tilde{\theta}^\alpha} = \overline{f^*\tilde{\theta}^\alpha}$ (still denoted by $\tilde{\theta}$, $\tilde{\theta}^\alpha$, $\overline{\tilde{\theta}^\alpha}$), we may assume that (2.1) and (2.2) possess the same family of solution curves on the common base space M. Since the chains are not defined for the directions in the maximal complex subspace $\theta = 0$ ($\tilde{\theta} = 0$) at each point, it follows that

$$\tilde{\theta} = \lambda\theta \quad \text{for some nonzero real smooth function } \lambda \tag{2.3}$$

Express $\tilde{\theta}^\alpha$ with respect to the base θ^β, $\theta^{\bar{\beta}}$, θ,

$$\tilde{\theta}^\alpha = U^\alpha_\beta\theta^\beta + V^\alpha_\beta\theta^{\bar{\beta}} + v^\alpha\theta \tag{2.4}$$

for some complex valued smooth functions U^α_β, V^α_β, v^α, $1 \le \alpha$, $\beta \le n$. Along a solution curve, substituting the first equations in (2.1) and (2.2) into (2.4) and equating the coefficients of θ using (2.3), we obtain

$$U^\alpha_\beta\sigma^\beta + V^\alpha_\beta\overline{\sigma^\beta} + v^\alpha = \tilde{\sigma}^\alpha\lambda. \tag{2.5}$$

Note that for each α, $1 \le \alpha \le n$, at each point, σ^α ranges over all complex numbers.

Differentiating (2.5) and equating the terms involving $\sigma^\beta\sigma^{\beta_1}\sigma^{\beta_2}$, $\sigma^\beta\sigma^{\beta_1}\sigma^{\bar{\beta}_2}$ and $\sigma^{\bar{\beta}}\sigma^{\beta_1}\sigma^{\bar{\beta}_2}$ (the expressions $d\sigma^\alpha$ and $d\tilde{\sigma}^\alpha$ in (2.1) and (2.2) were plugged in), we achieve the following algebraic equations:

$$(2.6)\quad \begin{aligned}
&\text{(i)} && U^\alpha_\beta B_{\beta_1\beta_2} + \text{terms interchanging any two of } \beta, \beta_1 \text{ and } \beta_2 = 0,\\
&\text{(ii)} && V^\alpha_\beta \overline{B}_{\beta_2\beta_1} + \text{terms interchanging any two of } \beta, \beta_1 \text{ and } \beta_2 = 0,\\
&\text{(iii)} && U^\alpha_\beta (C_{\beta_1\beta_2} - \lambda^{-1}A_{\beta_1\beta_2}) - \lambda^{-1}V^\alpha_{\beta_2} B_{\beta\beta_1} + \text{the term interchanging } \beta \text{ and } \beta_1 = 0,\\
&\text{(iv)} && V^\alpha_\beta (C_{\beta_1\beta_2} + \lambda^{-1}A_{\beta_1\beta_2}) + \lambda^{-1}U^\alpha_{\beta_1} \overline{B}_{\beta_2\beta} + \text{the term interchanging } \beta \text{ and } \beta_2 = 0,
\end{aligned}$$

where

$$\begin{aligned}
A_{\beta_1\beta_2} &= \sum_{\tilde\alpha_+=1}^{\tilde p} U^{\tilde\alpha_+}_{\beta_1}\overline{U^{\tilde\alpha_+}_{\beta_2}} - \sum_{\tilde\alpha_-=\tilde p+1}^{\tilde p+\tilde q=n} U^{\tilde\alpha_-}_{\beta_1}\overline{U^{\tilde\alpha_-}_{\beta_2}}\\
&\quad + \sum_{\tilde\alpha_+=1}^{\tilde p} V^{\tilde\alpha_+}_{\beta_2}\overline{V^{\tilde\alpha_+}_{\beta_1}} - \sum_{\tilde\alpha_-=\tilde p+1}^{\tilde p+\tilde q=n} V^{\tilde\alpha_-}_{\beta_2}\overline{V^{\tilde\alpha_-}_{\beta_1}},\\
B_{\beta_1\beta_2} &= \sum_{\tilde\alpha_+=1}^{\tilde p} U^{\tilde\alpha_+}_{\beta_1}\overline{V^{\tilde\alpha_+}_{\beta_2}} - \sum_{\tilde\alpha_-=\tilde p+1}^{\tilde p+\tilde q=n} U^{\tilde\alpha_-}_{\beta_1}\overline{V^{\tilde\alpha_-}_{\beta_2}},\\
C_{\beta_1\beta_2} &= \sum_{\alpha_+=1}^{p} \delta^{\alpha_+}_{\beta_1}\delta^{\alpha_+}_{\beta_2} - \sum_{\alpha_-=p+1}^{p+q=n} \delta^{\alpha_-}_{\beta_1}\delta^{\alpha_-}_{\beta_2}.
\end{aligned}$$

Now, suppose $V^{\alpha_0}_{\beta_0} \neq 0$ for some α_0, β_0 (at some point therefore a neighborhood of this point). By taking $\beta = \beta_1 = \beta_2 = \beta_0$ in (ii) of (2.6), we obtain

$$(2.7)\qquad B_{\beta_0\beta_0} = 0.$$

By (2.7), letting $\beta = \beta_1 = \beta_2 = \beta_0$ in (iv) of (2.6) gives

$$(2.8)\qquad A_{\beta_0\beta_0} = -\lambda C_{\beta_0\beta_0}.$$

If $U^{\alpha_1}_{\beta_0} \neq 0$ for some α_1, then taking $\beta = \beta_1 = \beta_2 = \beta_0$ in (iii) of (2.6) gives that $A_{\beta_0\beta_0} = +\lambda C_{\beta_0\beta_0}$. Since $C_{\beta_0\beta_0} = \pm 1$ and $\lambda \neq 0$, we get a contradiction to (2.8). Hence $U^\alpha_{\beta_0} = 0$ for all α. Now if $U^{\alpha_2}_\beta \neq$ for some α_2 and $\beta \neq \beta_0$ (the case $n = 1$ is trivial) then taking $\beta_1 = \beta_0$, $\beta_2 = \beta$ in (i) of (2.6) gives that $B_{\beta_0\beta} + B_{\beta\beta_0} = 0$. But since $U^\alpha_{\beta_0} = 0$ for all α, $B_{\beta_0\beta}$ must vanish. It follows that $B_{\beta\beta_0} = 0$.

Letting $\beta_1 = \beta_2 = \beta_0$ in (iii) of (2.6) and using (2.8) and vanishing of $B_{\beta_0\beta}$ and $B_{\beta\beta_0}$, we obtain $2C_{\beta_0\beta_0}U^\alpha_\beta = 0$. But $C_{\beta_0\beta_0} = \pm 1$ and $U^{\alpha_2}_\beta \neq 0$. So we get a contradiction. Therefore U^α_β must be zero for all α and β. But now (2.4) is reduced to $\tilde\theta^\alpha = V^\alpha_\beta \overline{\theta^\beta} + v^\alpha\theta$. Together with $\tilde\theta = \lambda\theta$, this means that f is a conjugate CR isomorphism. We are done. So we may assume $V^\alpha_\beta = 0$ for all α, β at the beginning. But that gives $\tilde\theta^\alpha = U^\alpha_\beta\theta^\beta + v^\alpha\theta$ which means that f is a CR isomorphism.

REMARK. The geometry of CR structure was generalized by N. Tanaka. In [**5**], he introduced the notion of a $\tilde G$-structure of type $\mathcal{M}$. Under certain condition, we may as well define some family of invariant curves as chains for CR geometry. And the analogous question as we answer for chains in this note can be asked for this class of geometries.

BIBLIOGRAPHY

1. D. Burns, Jr. and S. Shnider, *Real hypersurfaces in complex manifolds*, Several Complex Variables, Proc. Sympos. Pure Math., vol. 30, part 2, Amer. Math. Soc., Providence, R. I., 1977, pp. 141–168.
2. E. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes.* I, II, *Oeuvres* II, 2, pp. 1231–1304; *ibid.* III, 2, pp. 1217–1238.
3. S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.
4. H. Jacobowitz, *Chains in CR geometry*, J. Differential Geom. **21** (1985), 163–194.
5. N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan. J. Math. **2** (1976), 131–190.
6. K. Yano, *The theory of Lie derivatives and its applications*, North-Holland, Amsterdam, 1957.

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