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REPRESENTATION THEORY OF $U_1(H)$

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ABSTRACT. Mixed tensor representations of the inductive limit unitary group are studied according to factoriality, quasi-equivalence, and irreducibility

In the representation theory of inductive limit groups, wide classes of representations are studied via the dynamical system (X, G) attached to an AF C^*-algebra, even for nonlocally compact groups [**17**]. Questions of factoriality and equivalence are translated into problems of ergodicity and equivalence of probability measures on X. Recently, Kerov and Vershik [**9**] found that for the infinite symmetric group that a generalization of the Knuth correspondence [**11**] gave a map from an infinite product space to X which revealed the inherent multiplicative structure of the central ergodic measures on X. Such X were called projectivized.

With these ideas, we study the class of KMS representations of $U(\infty)$ introduced by Stratila and Voiculescu [**18**] who showed that these include the well-known gauge-invariant quasi-free states of the CAR algebra. Further, in [**6**], we established that these KMS representations contain the quasi-free states of the CCR algebra as well. A natural invariant approach to these representations is given by taking their continuous extension to $U_1(H)$. We give a brief summary of our results. Theorem 1 gives a factor condition which is useful for projectivized systems. In §2, we show that the space X for these $U(\infty)$ representations is projectivized by the Knuth correspondence. Our main result (Theorem 2) states that ergodicity on X is equivalent to ergodicity on the corresponding infinite product space, where the question was settled in [**1**]. Quasi-equivalence and irreducibility are examined in §4. In the final section, we observe that these irreducible representations correspond naturally to representations of the restricted unitary group which is important in applications [**13**, **15**].

Baker and Powers [**3**, **4**] studied more general questions about restrictions of states of UHF-algebras. It would be interesting to discover when their algebras are projectivized.

1. General factor condition. Let $A = \varinjlim A_n$ be a unital AF-algebra, with $A_0 = \mathbf{C} \cdot I$. By [**17**, Chapter I], there is a canonical dynamical system (X_A, G_A) attached to A, where $X = X_A$ is a compact Hausdorff space and $G = G_A$ is a discrete group of homeomorphisms of X. For simplicity, we assume that the partial multiplities of the embeddings of A_n into A_{n+1} are either 0 or 1. Then X is the projective limit of the finite sets X_n, with $X_n = (A_n)\hat{}$. It is useful to treat X as the space of infinite paths through the set of nodes $\bigcup_{n=0}^{\infty} X_n$.

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To describe probability measures on X, it is helpful to introduce more notation. Let $\mathrm{Path}(X_i, X_j)$ be the set of all paths which initiate in X_i and terminate in X_j, and let $m_{i,j}$ be a probability measure on it. We assume that the family $(m_{i,j})$ is multiplicative with respect to path concatenation; that is, if $S \in \mathrm{Path}(X_{i,j})$ and $T \in \mathrm{Path}(X_{j+1}, X_k)$, then $m_{i,k}(S * T) = m_{i,j}(S)m_{j+1,k}(T)$, where $*$ denotes path concatenation. Let m denote the resulting projective limit measure. Such measures m that arise from a multiplicative family will themselves be called *multiplicative.*

For $t_j \in X_{r(j)}$, $j = 1, \ldots, n$, let $C(t_1, \ldots, t_n)$ denote the cylinder set of all paths in X that pass through the nodes t_j, $j = 1, \ldots, n$. For $k \in X_r$, let $m(C(k)|p)$ denote the conditional probability of $C(k)$ given the cylinder set $C(p)$, where $p \in \mathrm{Path}(X_0, X_n)$. Finally, we observe that for any path $q \in \mathrm{Path}(X_i, X_j)$ that $m_{i,j}(q) = m(C(q))$.

THEOREM 1. *Let m be a G-quasi-invariant multiplicative probability measure on (X, G). Then m is G-ergodic if and only if for any path $p \in \mathrm{Path}(X_0, X_n)$,*

$$\lim_{r\to\infty} \sum \{|m(C(k)|p) - m(C(k))| : k \in X_r\} = 0.$$

PROOF. By [**17**, I.3.14], m is G-ergodic if and only if for every $f \in \mathscr{C}(X)$, $\varepsilon > 0$, there is an integer $r > 0$ so that for any G_r-invariant continuous function g, $|V(f,g)| \le \|g\|_\infty$, where $V(f,g) = \int f\,dm \int g\,dm - \int fg\,dm$. As in [**17**, p. 99], we may also assume that f is the characteristic function of a cylinder set $C(p)$, $p \in \mathrm{Path}(X_0, X_n)$, and that g depends only on the first N coordinates of x, with $n \le r \le N$. For such g, $g(x) = g(x' * y) = h(k, y)$, where $x' \in \mathrm{Path}(X_0, X_r)$ with terminal node k and $y \in \mathrm{Path}(X_{r+1}, X_N)$. Then

$$V(f,g) = \sum \{m(C(y))h(k,y)\,[m(C(k))m(C(p)) - m(C(p,k))] : \\ k \in X_r,\ y \in \mathrm{Path}(k, X_N)\}, \quad N > r;$$

and if $N = r$, the factor $m(C(y))$ is omitted in the above sum. Here $\mathrm{Path}(k, X_N)$ is the set of all paths which initiate at the node k and terminate in X_N. If

$$\lim_{r\to\infty} \sum \{|m(C(k))m(C(p)) - m(C(p,k))| : k \in X_r,\ y \in \mathrm{Path}(k, X_N)\} = 0,$$

then m is G-ergodic. Conversely, if m is G-ergodic, then the above limit must be zero. To see this, choose g suitably with $N = r$ and with values ± 1.

The condition of the theorem follows easily since

$$m(C(k,p)) = m(C(k)|p)m(C(p)).$$

Natural examples of multiplicative measures occur if the Bratteli diagram of A is *projectivized* in the sense of Kerov and Vershik [**9**]. This means there is a finite set L and a continuous map T from $L^\infty = \prod_{n=1}^\infty L$ onto X which is compatible with the canonical projection maps of these two projective limit spaces. Then $m = m' \circ T^{-1}$ is a multiplicative measure on X if m' is any infinite product of probability measures on L. For the infinite symmetric group $S(\infty)$, the G-invariant ergodic probability measures correspond to the infinite product of identical measures [**9**]. It would be interesting to discover if the results of Baker and Powers [**4**] can be interpreted in this context.

2. Tensor product algebra T_k. Let H be a separable infinite-dimensional complex Hilbert space with fixed orthonormal basis $\mathscr{E}$. Let $U_1(H)$ be the group of unitary operators which are trace-class perturbations of the identity, and let $U(\mathscr{E})$ be the inductive limit unitary group taken with respect to the basis $\mathscr{E}$.

By [**17**], the primitive quotient of $C^*(U(\mathscr{E}))$ that supports antisymmetric tensor representations can be identified with $\mathscr{I}(H)$, the gauge-invariant subalgebra of the CAR algebra over H. Let $\overline{\rho} : U_1(H) \to \mathscr{I}(H)$ denote this embedding. For $0 \leq A \leq I$, the positive definite function $p_A(V) = \det(I - A + AV)$ on $U_1(H)$ can be identified with the quasi-free state $\omega_{A'}$ on $\mathscr{I}(H)$, where $p_A(V) = \omega_{A'} \circ \overline{\rho}(V)$, $V \in U_1(H)$, and $A' = I - A$. Next, consider Hilbert spaces H_j, $j = 1, \ldots, k$, with $H_j \simeq H$, orthonormal bases $\mathscr{E}_j$, and embeddings ρ_j of $U_1(H_j)$ into $\mathscr{I}(H_j)$. Also, let $\rho = \rho_1 \times \cdots \times \rho_k : U(\mathscr{E}_1) \times \cdots \times U(\mathscr{E}_k) \to \mathscr{I}_k = \mathscr{I}(H_1) \otimes \cdots \otimes \mathscr{I}(H_k)$. Note that $\bigotimes_1^k \mathscr{I}(H_j) \subset \mathscr{I}(\bigoplus_1^k H_j)$, so that the product $p_A(V_1) \cdots p_A(V_k) = (\omega_{\tilde{A}'} \circ \rho)(V_1, \ldots, V_k)$, where $\tilde{A}'$ is the amplication of A' from H to $\bigoplus_1^k H_j$.

Let T_k denote the fixed point algebra of the natural action of $\mathbf{Z}_k$ on the tensor product $\bigotimes_1^k \mathscr{I}(H_j)$ (when $k = 2$, the action is just the "flip"). Then $\rho(V_1, \ldots, V_k) \in T_k$ if and only if $V_1 = \cdots = V_k$. Hence, if π is a representation of $U_1(H)$ in the antisymmetric tensors, $\bigotimes_1^k \pi$ is a representation of T_k. Let P_k be the primitive quotient of $C^*(U(\mathscr{E}))$ by the primitive ideal with upper signature $(k, k, \ldots)$ and zero lower signature. Because of the natural embedding of P_k into T_k, P_k can be identified as a subalgebra of the CAR algebra over $\bigoplus_1^k H_j$.

The dynamical system (X_I, G_I) attached to $\mathscr{I}_k$ can be identified:

$$X_I = \prod_1^\infty \{1, \ldots, 2^k\} \quad \text{and} \quad G_I = S(\infty).$$

Another useful realization is given by letting X_I be the projective limit of $(X_I)_n$, where $(X_I)_n$ is the space of $(0,1)$-matrices with n rows and k columns. If $L = \{0,1\}^k$, then $X_I = L^\infty$. For the dynamical system (X_P, G_P) attached to P_k, $(X_P)_n$ can be realized as the space of generalized Young tableaux with at most k columns and n entries [**11**]. The canonical projection of $(X_P)_{n+1}$ onto $(X_P)_n$ is given by the DELETE* algorithm of Knuth [**11**]. Then, the Knuth correspondence between elements of $(X_I)_n$ and pairs of conjugate generalized Young tableaux yields a map T of X_I onto X_P which implements a projectivization.

For the remainder of the paper we will adopt the convention of freely identifying the positive definite function $(p_A)^k$ with the corresponding state on either $\mathscr{I}_k, T_k$, or P_k.

3. Factor condition for $(p_A)^k$, where A is diagonal and $0 < A < I$. Let $0 < A < I$ be a diagonal operator with respect to the orthonormal basis $\mathscr{E} = \{e_j\}$ and with corresponding eigenvalues $\{p_j\}$. Set $q_j = 1 - p_j$. Then the positive definite function $(p_A)^k$ induces the product measure m' on $X_I = L^\infty$. Here, $m' = \bigotimes_1^\infty m_j$, where m_j is the probability measure on $L = \{0,1\}^k$ determined by the generating function $(p_j + q_j z_1) \cdots (p_j + q_j z_k)$. The measure $m = m' \circ T^{-1}$ on X_P is determined uniquely by its values on the cylinder sets of paths that pass through the node λ, where λ is the signature (or shape, in the terminology of [**11**]) of an irreducible representation of $U(N)$. Because of the well-known symmetric function identity [**11**, §7; **16**, 9.2], the m-measure of this cylinder set is $(p_1 \cdots p_N)^k s_\lambda(p_1/q_1, \ldots, p_N/q_N)$,

where s_λ is the Schur function indexed by λ. (See [**16**] for a discussion of Schur functions.) We now can state the main result of this paper.

THEOREM 2. *With the above notation, the product measure m' is G_I-ergodic on X_I if and only if the induced measure $m = m' \circ T^{-1}$ on X_P is G_P-ergodic.*

To prove that the ergodicity of m' implies the ergodicity of m, we need to introduce two polynomial norms. Given the polynomial

$$F(z_1, \dots, z_k) = \sum a(i_1, \dots, i_k) z_1^{i_1} \cdots z_k^{i_k} = \sum a_\alpha z^\alpha$$

(standard multi-index convention), let $|F|_1$ be the usual l_1-norm: $\sum |a_\alpha|$. In addition, if F is also symmetric, we define the weaker norm $|F|_s$. Recall that the vector space $\mathrm{Sym}(z_1, \dots, z_k)$ of all symmetric polynomials admits an inner product $\langle \cdot, \cdot \rangle$ such that the Schur functions s_λ, $\lambda \in \mathbf{P}_k$ (= set of all partitions with no more than k nonzero terms), form an orthonormal basis. So, if $F = \sum \{c_\lambda s_\lambda : \lambda \in \mathbf{P}_k\}$, define $|F|_s$ to be the usual l_1-norm with respect to this expansion; that is, $|F|_s = \sum \{|c_\lambda| : \lambda \in \mathbf{P}_k\}$.

Following the reasoning of [**17**, Chapter IV] and by applying our Theorem 1, we have that m' is G_I-ergodic if and only if, for any $\alpha : i_1, \dots, i_k \geq 0$, $i_1 + \cdots + i_k = n$,

$$\lim_{r \to \infty} \left| \prod_{j=n+1}^{r} \prod_{i=1}^{k} (p_j + q_j z_i) \cdot \left(z^\alpha - \prod_{j=1}^{n} \prod_{i=1}^{k} (p_j + q_j z_i) \right) \right|_1 = 0.$$

To conclude the analogous result for P_k, we must first recall two fundamental Schur function identities:

(a) *Littlewood-Richardson Rule.* For any $f \in \mathrm{Sym}(z_1, \dots, z_k)$, $\langle s_\lambda f, s_\nu \rangle = \langle f, s_{\nu/\lambda} \rangle$, where $s_{\nu/\lambda}$ denotes the skew-Schur function indexed by the skew partition ν/λ [**16**].

(b) *Branching Rule.* By treating Schur functions as characters of the corresponding unitary group, we have $s_\nu | U(n) \times U(k-n) = \sum \{s_\lambda \times s_{\nu/\lambda} : \lambda \in \mathbf{P}_n\}$ [**10**].

As above, we again have by a straightforward application of Theorem 1: $m = m' \circ T^{-1}$ is G_P-ergodic if and only if, for any $\lambda \in \mathbf{P}_k$,

$$\lim_{r \to \infty} \left| \prod_{j=n+1}^{r} \prod_{i=1}^{k} (p_j + q_j z_i) \left(s_{\lambda'}(z_1, \dots, z_k) - \prod_{j=1}^{n} \prod_{i=1}^{k} (p_j + q_j z_i) \right) \right|_s = 0,$$

where λ' denotes the conjugate of λ. Since the l_1-norm dominates the s-norm and the Schur function $s_{\lambda'}$ is a sum of monomials, the ergodicity of m' implies the ergodicity of m.

To establish the converse, we need to invoke the following theorem of Aldous and Pitman [**1**]:

THEOREM. *m' is G_I-ergodic if and only if either:*
(a) *for any $B \subset L$, $\sum_{j=1}^{\infty} \inf(m_j(B), m_j(B^c)) = 0, \infty$; or*
(b) $\sum_{j=1}^{\infty} p_j^i (1 - p_j)^{k-i} = 0, \infty$, $0 \leq i \leq k$.

We note that the sum in (b) can never be 0 since $0 < p_j < 1$.

We adapt the reasoning of [**18**, p. 89]. Realize $(X_I)_n$ as the space of $(0,1)$ $n \times k$ matrices, and let T_n denote the Knuth transform between $(X_I)_n$ and $(X_P)_n$. Since the shape of the generalized Young tableau $T_n(x)$, $x \in (X_I)_n$, is determined by the row sums of x [**11**, **16**], $T_n(S)$, $S \subset (X_I)_n$, will be $(G_P)_n$-invariant if S is closed under the independent actions of the symmetric group $S(k)$ on any row of x in S. Moreover, for such S, $S = T_n^{-1}(T_n(S))$.

Suppose m' is not G_I-ergodic. Then there is a subset B of L such that $0 < \sum \inf(m_j(B), m_j(B^c)) < \infty$. By construction, each measure m_j is $S(k)$-invariant. As a consequence, we can require that B be a proper $S(k)$-invariant subset of L. Let $I \subset \mathbf{Z}^+$ such that $\sum\{m_j(B) \colon j \in I\}$ and $\sum\{m_j(B^c) \colon j \in I^c\}$ are both convergent. Let $Z = \{x \in L^\infty \colon x_n \in B^c,\ n \in I; x_n \in B,\ n \in I^c\}$. Then $m'(Z) > 0$. For $d \in \mathbf{Z}$, let

$$Z_d = \{x \in L^\infty \colon |\{n \in I \colon x_n \in B\}| - |\{n \in I^c \colon x_n \in B^c\}| = d\}.$$

Then, we can check that $m'(Z_d) > 0$, $-|I^c| \le d \le |I|$, since $m'(Z) > 0$. Moreover, the Z_d's are mutually disjoint and are invariant under the action of $S(k)$ on any row. Hence, m fails to be G_P-ergodic since $m'(Z_d) = m(T(Z_d))$.

In view of the theorem of Aldous and Pitman, we can restate Theorem 2 as

THEOREM 3. *Let $0 < A < I$ be diagonal with eigenvalues $\{p_j\}$. Then $(p_A)^k$ is factorial if and only if $\sum_{j=1}^\infty p_j^i (1-p_j)^{k-i} = \infty$, $0 \le i \le k$.*

4. Quasi-equivalence. We first state a lemma which occurs throughout §3 of [**16**]. A similar result is given in [**2**]; a discussion of this problem also appears in [**8**, p. 202].

LEMMA 4. *Let $0 \le S, T \le I$ be such that $S^{1/2} - T^{1/2}$ and $(I-S)^{1/2} - (I-T)^{1/2}$ lie in $\mathscr{C}_2$, the ideal of Hilbert-Schmidt operators on H. Then there is a path (A_t) which is continuous in the Hilbert-Schmidt norm such that $A_0 = S$, $A_1 = T$, and $0 \le A_t \le I$. Moreover, if S and T are also projections with $cd(S,T) = \dim(S - s(STS)) - \dim(T - s(TST)) = 0$ (here, $s(X)$ is the support projection of X), then the path (A_t) can be chosen to lie in the space of projections. If ω_t is the quasi-free state indexed by the operator A_t, then the path (ω_t) is norm continuous in the state space of the CAR algebra over H.*

PROPOSITION 5. *Let $0 \le A \le I$ and $\mathrm{Tr}(A^i(I-A)^{k-i}) = \infty$, $i = 0, \ldots, k$. Then $(p_A)^k$ is factorial on T_k.*

PROOF. The same method as in [**18**, §3] holds here as well. We approximate A by a sequence of invertible diagonalizable operators (A_n), $0 < A_n < I$, such that $\|A^{1/2} - A_n^{1/2}\|_2 \to 0$ and $\|(I-A)^{1/2} - (I-A_n)^{1/2}\|_2 \to 0$, where $\|X\|_2$ denotes the Hilbert-Schmidt norm of X. Then the corresponding quasi-free states ω_n converge in norm to the state that corresponds to $(p_A)^k$. But the set of all factor states is norm closed.

THEOREM 6. *Let $0 \le A, B \le I$, $\mathrm{Tr}(A^i(I-A)^{k-i}) = \infty$, $\mathrm{Tr}(B^i(I-B)^{k-i}) = \infty$, $i = 0, \ldots, k$. Then $(p_A)^k$ is quasi-equivalent to $(p_B)^k$ if and only if $A^{1/2} - B^{1/2}$ and $(I-A)^{1/2} - (I-B)^{1/2}$ both lie in $\mathscr{C}_2$.*

PROOF. Because there is a faithful conditional expectation Φ_1 of $\mathscr{I}_k$ onto T_k, quasi-equivalence of states, say p_1 and p_2, on T_k implies that $p_1 \circ \Phi_1$ and $p_2 \circ \Phi_1$

are quasi-equivalent on $\mathscr{I}_k$ [**18**, 2.7]. But $\mathscr{I}_k$ is the fixed point algebra under the gauge action of $S^1 \times \cdots \times S^1$ (k times) on the CAR algebra A over $\bigoplus_1^k H_j$. Hence, there is a faithful conditional expectation of A onto $\mathscr{I}_k$. In the notation of §2, quasi-equivalence of $(p_A)^k$ and $(p_B)^k$ on T_k implies that the quasi-free states $\omega_{\tilde{A}'}$ and $\omega_{\tilde{B}'}$ are quasi-equivalent on A. By [**14**], this implies necessity. To argue for sufficiency, apply Lemma 4 to obtain a norm continuous path (ω_t) of quasi-free states with $(p_A)^k = \omega_0$, $(p_B)^k = \omega_1$. By choosing a partition $0 = t_0 < \cdots < t_n = 1$ of $[0, 1]$, we can construct a chain of quasi-equivalent factor states connecting $(p_A)^k$ and $(p_B)^k$.

For the remainder of this section, we study the situation in which $\mathrm{Tr}(A^i(I - A)^{k-i})$ may be finite or zero. We first consider the case that A is a projection.

For cyclic representations (r, v) and (r', v') of a group G with cyclic vectors v and v', define their *Cartan product* $(r * r', v \otimes v')$ to be the cyclic subrepresentation of $r \otimes r'$ with cyclic vector $v \otimes v'$. If $r, r' \in U(N)\hat{}$ and the cyclic vectors are also highest weight vectors, then $r * r'$ is irreducible.

We shall be considering a standardized class of direct limit representations π of $U(\infty)$ where $(\pi, v_\infty) = \varinjlim(\pi_n, v_n)$, $\pi_n \in U(n)\hat{}$; v_n's are highest weight vectors which are consistently embedded, and $v_\infty = \varinjlim v_n$. For such representations, it is elementary to prove

LEMMA. *If (π, v_∞) and (π', v'_∞) are two such irreducible representations of $U(\infty)$, then their Cartan product is also irreducible.*

A key example of such irreducible representations is given in [**18**] where it is shown that $p_E(V)$ has this form if E is a projection. To state the behavior of their products, we introduce some notation. Let $\mathbf{E} = \{E_j\}_{j=1}^n$ be a set of projections and $\alpha = \{\alpha(j)\}_{j=1}^n$ be positive integers. We now form $p(\mathbf{E}, \alpha)(V) = \prod_{j=1}^n p_{E_j}(V)^{\alpha(j)}$.

PROPOSITION 7. (a) *$p(\mathbf{E}, \alpha)$ is irreducible;*

(b) *given $p(\mathbf{E}, \alpha)$ and $p(\mathbf{E}', \alpha)$, then they are unitarily equivalent if and only if $\|E_j - E'_j\|_2 < \infty$ and $cd(E_j, E'_j) = 0$, $j = 1, \ldots, n$.*

PROOF. (a) This follows directly from the above Lemma.

(b) The proof of Theorem 6 extends to this context with the following observation. Form the direct sums A and A' of the sets of projections $\mathbf{E}$ and $\mathbf{E}'$, counting multiplicities, on $\bigoplus_{j=1}^M H_j$, where $M = \alpha(1) + \cdots + \alpha(n)$. Then $\omega_{\tilde{A}}$ and $\omega_{\tilde{A}'}$ are unitarily equivalent: on $\mathscr{I}_M$ if and only if they are equivalent on the full CAR algebra over $\bigoplus_{j=1}^M H_j$. We have (b) by [**14**].

We now give the general factor result.

THEOREM 8. *Let $0 \le A \le I$. Then $(p_A)^k$ is factorial iff $\mathrm{Tr}(A^i(I - A)^{k-i}) = 0$, ∞, for $i = 0, 1, \ldots, k$.*

PROOF. If $\mathrm{Tr}(A^i(I - A)^{k-i}) = 0$, for some i between 0 and k, then A must be a projection since $0 \le A \le I$. Hence, we need consider only the necessity of the trace condition. This argument is analogous to the proof of the necessity of Theorem 2. So, assume that $(p_A)^k$ is factorial and that for index i the trace of $A^i(I - A)^{k-i}$ is nonzero and finite.

Since $0 \leq A \leq I$, $A^i(I-A)^{k-i}$ must be compact. So, the essential spectrum of A consists of, at most $\{0,1\}$ while the other points of the spectrum are isolated eigenvalues of finite multiplicity. Choose an orthonormal basis $\mathscr{E}$ of eigenvectors with corresponding eigenvalues $\{p_j\}$. With respect to $\mathscr{E}$, form the dynamical systems for $\mathscr{I}_k$ and P_k. Note that the product measure $m' = \bigotimes_1^\infty m_j$ on X_I has the form: $m_j = \bigotimes_1^k \mu_j$, $\mu_j(\{0\}) = p_j$ and $\mu_j(\{1\}) = 1 - p_j = q_j$. Unlike the case in §3, p_j may be either 0 or 1.

Choose $I_0 = \{n \in \mathbf{Z}^+ \colon \mu_n(\{0\}) = 1\}$, $I_1 = \{n \in \mathbf{Z}^+ \colon \mu_n(\{1\}) = 1\}$, $I = \mathbf{Z}^+ \backslash (I_0 \cup I_1)$. Let $a_n = (0,\ldots,0) \in L$, $b_n = (1,\ldots,1) \in L$. Let $S = \{x \in L^\infty \colon x_n = a_n,\ n \in I_0; x_n = b_n,\ n \in I_1\}$. If $|I| < \infty$, we can construct in an elementary combinatorial fashion disjoint subsets of $X_I = L^\infty$ which are closed under the action of $S(k)$ on any row of $x \in X_I$. Just as in §3, this contradicts the ergodicity of $m = m' \circ T^{-1}$. If $|I| = \infty$, then we choose $B \subset I$ as in the argument of necessity of Theorem 2. That argument extends to this setting by intersecting the sets Z and Z_d with S.

REMARKS. (1) By multiplying by powers of the determinant, all of our results can be extended to positive definite functions of the form $p_A(V)^k p_A(V^{-1})^m$. (2) In [**5**], we constructed a factor representation of $U(H)_2$ via the positive definite function p_A with $\mathrm{Tr}(A(I-A)) = \infty$ but $A \in \mathscr{C}_2$. By Theorem 8, $(p_A)^2$ is no longer factorial.

5. Applications. (A) Let $H = H_+ \oplus H_-$ with $\dim(H_+) = \dim(H_-) = \infty$, and let H_+, H_- have orthonormal bases $\mathscr{E}_+, \mathscr{E}_-$, respectively. Then the inductive limit group $U(\mathscr{E})$ contains both the inductive limit symplectic group $\mathrm{Sp}(\infty)$ and the special orthogonal group $SO(\infty)$. Moreover, there is a natural copy of $U(\infty)$ that lies as a subgroup in both these classical groups. Let E be a projection of H with range contained in H_+; let P_+ be the projection of H onto H_+. Then the positive definite function $(p_E)^k$ corresponds to an irreducible representation on either $U(\infty)$ or $U(\mathscr{E})$. Thus, by Proposition 7, we have

THEOREM 9. *Let E and F be projections such that $E, F \leq P_+$. Then $(p_E)^k$ induces an irreducible representation on either* $\mathrm{Sp}(\infty)$ *or $SO(\infty)$ by restriction. Moreover, $(p_E)^k$ is unitarily equivalent to $(p_F)^k$ if and only if $\|E - F\|_2 < \infty$.*

(B) The restricted unitary group U_r consists of all unitary operators V on H such that $[V, P_+] \in \mathscr{C}_2$. For $V \in U(H)$, write $a(V)$ for $P_+ V P_+$. Then, as in [**15**, **12**], set $\tilde{U}_r = \{(V,q) \in U_r \times U(H_+) \colon a(V)q^{-1} \in U_1(H_+)\}$, and let $\hat{U}_r = U_r / SU_1(H_+)$, the affine extension of U_r. The basic representation of $\hat{U}_r$ has spherical function f which is given by the $SU_1(H_+)$-invariant function $\tilde{U}_r \colon f(V,q) = \det(a(V)q^{-1})$. Note that on the subgroup $U_1(H) \times U_1(H_+)$ of $\tilde{U}_r$, $f(V,q) = p_{P_+}(V)\det(q^{-1})$. So, it is natural to consider the more general collection of positive definite functions $f(V,q)^{n(0)} p_{F_1}(V)^{n(1)} \cdots p_{F_m}(V)^{n(m)}$, where $n(j) \in \mathbf{Z}$, $j = 0,1,\ldots,m$, such that the range of F_j is given by the first $n(j)$ elements of the basis $\mathscr{E}_+$. These functions naturally correspond to the coadjoint orbits of $(U_r)_0$ in the predual of the Lie algebra for U_r [**13**].

It would be interesting to try to combine the techniques of [**5**, **7**, **12**] to give a holomorphic induction realization of the corresponding irreducible representations.

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