MORE ON THE DIFFERENTIABILITY OF CONVEX FUNCTIONS

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ABSTRACT. Let $C$ be a closed, convex set in a topological vector space $X$ such that $NS(C)$, the set of its nonsupport points, is nonempty (this is always the case if $X$ is Banach separable; if $X$ is Fréchet, $NS(C)$ is residual in $C$). If $X$ is normed, we prove that any locally Lipschitz, convex real function $f$ on $C$ is subdifferentiable on $NS(C)$. If in addition $X$ is Banach separable, we prove that $f$ is smooth on a residual subset of $NS(C)$.

Let $X$ be a topological vector space and $X^*$ be its topological dual. Let $C$ be a closed convex subset of $X$. Recall that a point $x_0$ of $C$ is called a support point of $C$ if there exists a nonzero $f$ in $X^*$ such that $f(x) < f(x_0)$ for any $x$ in $C$. We denote by $S(C)$ the set of all support points of $C$ and by $NS(C)$ the set of all nonsupport points of $C$. If $NS(C)$ is nonempty, then it is convex and has the following property: for any $x_0$ in $NS(C)$ and $x$ in $C$, the segment $[x_0, x]$ is contained in $NS(C)$; in particular $NS(C)$ is dense in $C$. As a matter of fact a stronger result holds if $X$ is Fréchet: $NS(C)$ is a residual subset of $C$. To see this it is sufficient to show that $S(C)$ is a set of first category in $C$. This last assertion is an immediate consequence of Theorem 1 in [5] (asserting that $S(C)$ is an $F_\sigma$) and of the fact that $NS(C)$ is dense in $C$.

If $X$ is a Banach space, it is well known (Bishop-Phelps theorem, see for example [3]) that $S(C)$ is dense in the boundary of $C$. As a matter of fact, if $C$ has nonempty interior, then $NS(C)$ is exactly the interior of $C$ and $S(C)$ is exactly the boundary of $C$ [3, p. 64]. Notice also that if $C$ has empty interior, then $NS(C)$, if nonempty, is neither open nor closed in $C$ (as $NS(C)$ and $S(C)$ are both dense in $C$). It is obvious that if $C$ is contained in a closed hyperplane, then $NS(C)$ is empty. Even if $C$ is not contained in a closed hyperplane, it may happen that $NS(C)$ is empty [4, Example 3, p. 460]. However, if $X$ is separable and $C$ is not contained in a closed hyperplane, then $NS(C)$ is nonempty [2, p. 655]. It follows that, under this assumption on $X$, $NS(C) \neq \emptyset$ iff $cl(aff(C)) = X$ (as usual $aff(C)$ denotes the smallest affine subspace containing $C$ and $cl$ stands for the topological closure).

Let $X$ be a Hausdorff locally convex space and $C$ be a closed convex subset of $X$ such that $NS(C) \neq \emptyset$. Let

$$\text{cone}_x(C) = \{ y \in X; x + ty \in C \text{ for some } t > 0 \}$$

and

$$T_x(C) = cl(\text{cone}_x(C)).$$

Notice that $x \in NS(C)$ if and only if $T_x(C) = X$. 

We can now state and prove our main results.

**Theorem 1.** Let $C$ be a closed convex subset of the normed, vector space $X$ and $f: C \to R$ be convex and locally Lipschitz on $NS(C)$. Then

(i) $f$ is subdifferentiable on $NS(C)$;

(ii) for any $x$ in $NS(C)$ the set of subgradients of $f$ at $x$ is weak*-compact.

**Proof.** Let $z \in NS(C)$ and $v \in \text{cone}_x(C)$. Let $t > 0$ be such that $x + tv \in C$. Define $d_{x,v} : (0, t) \to R$ by

$$d_{x,v}(h) = \frac{f(x + hv) - f(x)}{h}.$$

$d_{x,v}$ is increasing (as $f$ is convex) and bounded (as $f$ is locally Lipschitz on $NS(C)$). It follows that $\lim_{h \to 0} d_{x,v}(h)$ exists and is finite; denote it $f'(x; v)$. Since $f$ is convex, $f'(x; v) : \text{cone}_x(C) \to R$ is sublinear. The fact that $f$ is locally Lipschitz on $NS(C)$ implies that $f'(x; )$ is Lipschitz on $\text{cone}_x(C)$. Indeed there exist $\varepsilon_0, M > 0$ such that $f$ is Lipschitz on $B(x, \varepsilon_0) \cap NS(C)$ with Lipschitz constant $M$. Let $v, w \in \text{cone}_x(C)$ and $h$ sufficiently small such that $x + hv, x + hw \in B(x, \varepsilon_0) \cap NS(C)$. Then

$$\left| \frac{f(x + hv) - f(x + hw)}{h} \right| \leq M\|v - w\|.$$

As a consequence, there exists a unique continuous $F(x; ) : X \to R$ extending $f'(x; )$ (recall that $T_x C = X$). Clearly $F(x; )$ is sublinear and continuous (in fact Lipschitz with Lipschitz constant $M$), $F(x; 0) = 0$ and for any $y \in C$, $F(x; y - x) \leq f(y) - f(x)$. Assertion (i) follows now from the following remarks: (1) any subgradient of $F(x; )$ at zero is a subgradient of $f$ at $x$; (2) the set of subgradients of $F(x; )$ at zero is nonempty. The proof of assertion (ii) is similar to the proof of Theorem B, p. 84 in [3].

**Theorem 2.** Let $X$ be a separable Banach space, $C$ be a closed, convex subset of $X$ and $f: C \to R$ be convex and locally Lipschitz on $NS(C)$. Then $f$ is smooth on a residual subset of $NS(C)$.

**Proof.** We use the notation introduced above. Recall that for each $x \in NS(C)$ and $v \in X$, $-F(x; -v) \leq F(x, v)$. Since $X$ is separable there exists a countable dense subset $(v_n)_n$ of $X$. For any positive integers $m$ and $n$ let

$$A_{m,n} = \{ x \in NS(C); F(x; v_n) + F(x; -v_n) \geq 1/m \}$$

and let $A = \bigcup A_{m,n}$. We shall prove that each $A_{m,n}$ is closed in $NS(C)$ and has empty interior in $NS(C)$. Then for any $x \in NS(C) \setminus A$, $F(x; v) = -F(x; -v)$, which implies that $F(x; )$ is linear and therefore is a gradient of $f$ at $x$.

Let $v \in X$ and let

$$A_{m,v} = \{ x \in NS(C); F(x; v) + F(x; -v) \geq 1/m \},$$
Let us prove that each $A_{m,v}$ is closed in $NS(C)$. Assume the contrary: there exists a sequence $(x_n)_n$ in $A_{m,v}$ such that $\lim x_n = x_0 \in NS(C)$, but $x_0 \notin A_{m,v}$; that is, $F(x_0; v) + F(x_0; -v) < 1/m$. Let $\varepsilon > 0$ be such that

$$F(x_0; v) + F(x_0; -v) = 1/m - \varepsilon.$$  

As $T_{x_0}(C) = X$, there exist $u', u'' \in \text{cone}_{x_0}(C)$ such that

$$\|v - u'\| < \varepsilon/16M \quad \text{and} \quad \|v + u''\| < \varepsilon/16M$$

($M$ being the Lipschitz constant of $f$ on $B(x_0, \varepsilon_0)$). We have

$$|F(x_0; v) - F(x_0; u')| \leq M\|v - u'\| < \varepsilon/16$$

and

$$|F(x_0; -v) - F(x_0; u'')| \leq M\|v + u''\| < \varepsilon/16.$$  

It follows that $F(x_0; u') + F(x_0; u'') < 1/m - 7\varepsilon/8$; that is

$$f'(x_0; u') + f'(x_0; u'') < 1/m - 7\varepsilon/8$$

and therefore, for some $t > 0$, we have

$$f'(x_0; u') + f'(x_0; u'') < 1/m - 7\varepsilon/8.$$  

Let $v'_n = u' + (x_0 - x_n)/t$ and $v''_n = u'' + (x_0 - x_n)/t$, $n \in N$. Then $x_0 + tu' = x_n + v_n'$ and $x_0 + tu'' = x_n + v_n''$, implying that $v_n', v_n'' \in \text{cone}_{x_n}(C)$. Now (2) becomes

$$\frac{f(x_n + tu'_n) - f(x_n)}{t} + \frac{f(x_n + tu''_n) - f(x_n)}{t} + \frac{2(f(x_n) - f(x_0))}{t} \leq \frac{1}{m} - \frac{7}{8}\varepsilon,$$  

hence

$$f'(x_n; u'_n) + f'(x_n; u''_n) < \frac{1}{m} - \frac{7}{8}\varepsilon - \frac{2(f(x_n) - f(x_0))}{t}.$$  

For large $n$, $|f(x_n) - f(x_0)| < t\varepsilon/16$ (since $\lim x_n = x_0$); so

$$F(x_n; u'_n) + F(x_n; u''_n) < 1/m - 3\varepsilon/4.$$  

Since $x_n \in A_{m,v}$, one has $F(x_n; v) + F(x_n; -v) \geq 1/m$, $n \in N$. For large $n$, $x_n \in B(x_0, \varepsilon_0) \cap NS(C)$ and therefore

$$\|v - v'_n\| \leq \|v - u'\| + \|u' - v'_n\| < \frac{\varepsilon}{16M} + \frac{\|x_n - x_0\|}{t}$$

$$< \frac{\varepsilon}{16M} + \frac{\varepsilon}{16M} = \frac{\varepsilon}{8M}$$

(take $n$ large enough such that $\|x_n - x_0\| < t\varepsilon/16M$). Similarly

$$\|v + v''_n\| \leq \|v + u''\| + \|u'' - v''_n\| < \varepsilon/8M.$$  

At this point it is important to observe that $F(x; )$ is Lipschitz on $X$ with the same Lipschitz constant $M$ for all $x \in B(x_0, \varepsilon_0) \cap NS(C)$, hence

$$|F(x_n; v) - F(x_n; u'_n)| \leq M\|v - u'_n\| < \varepsilon/8$$

and

$$|F(x_n; -v) - F(x_n; u''_n)| \leq M\|v + u''_n\| < \varepsilon/8.$$  

Consequently

$$F(x_n; v'_n) + F(x_n; v''_n) > F(x_n; v) + F(x_n; -v) - \varepsilon/4$$

for large $n$. As $x_n \in A_{m,v}$ this is in conflict with (3), proving that $x_0 \in A_{m,v}$.

Assume now that the interior of $A_{m,v}$ in $NS(C)$ is nonempty. Let $x_0 \in \text{int}_{NS(C)}(A_{m,v})$; then $B(x_0, \varepsilon) \cap NS(C) \subset A_{m,v}$ for some $\varepsilon > 0$. For $x \in B(x_0, \varepsilon) \cap NS(C)$, $F(x; v) + F(x; -v) \geq 1/m$. As $T_{x_0}(C) = X$, there exists $v' \in \text{cone}_{x_0}(C)$ such that $\|v - v'\| < 1/2mM$ and $x_0 + t_0v' \in C$ for some $t_0 > 0$. We have $x_0 + tv' \in B(x_0, \varepsilon) \cap NS(C)$ for $t < t_0$ sufficiently small; but then $[x_0, x_0 + tv'] \subset B(x_0, \varepsilon) \cap NS(C)$. Define $g: (0, t) \to R$ by $g(s) = f(x_0 + sv')$. Since $g$ is convex, there exists $s_0 \in (0, t)$ such that $g'_+(s_0) = g'_-(s_0)$ ($g'_+, g'_-$ denoting as usual the right and left derivatives of $g$ at $s_0$). But

$$g'_+(s_0) = f'(x_0 + s_0v'; v')$$

and

$$g'_-(s_0) = -f'(x_0 + s_0v'; -v').$$

Thus $f'(x_0 + s_0v'; v') = -f'(x_0 + s_0v'; -v')$, or $F(x_0 + s_0v'; v') + F(x_0 + s_0v'; -v') = 0$ contradicting the fact that $x_0 + s_0v' \in A_{m,v}$. Thus, the interior of $A_{m,v}$ in $NS(C)$ is empty.

**COROLLARY.** Let $f: C \to R$ be convex and locally Lipschitz on $NS(C)$. Then the set of those $x$ in $NS(C)$ at which $f$ fails to be smooth cannot contain any closed convex subset $K$ such that $\text{cl}(\text{aff } K) = X$.

**REMARK.** If the interior of $C$ is nonempty, the same conclusions hold, but with the apparently weaker assumption that $f$ is lower semicontinuous on $\text{int}(C)$ (see for example [2, p. 167]). However, in this case the semicontinuity of $f$ is equivalent with $f$ being locally Lipschitz on $\text{int}(C)$. Without any assumption on the interior of $C$, it is known that the lower semicontinuity of $f$ implies its subdifferentiability on a dense subset of $C$ (if $X$ is Banach; see [1]). In our case, by imposing stronger conditions on $f$ we obtain its subdifferentiability on the whole $NS(C)$ and its smoothness on a residual subset of $NS(C)$, and hence of $C$.

**REFERENCES**


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