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MORE ON THE DIFFERENTIABILITY OF CONVEX FUNCTIONS

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ABSTRACT. Let C be a closed, convex set in a topological vector space X such that $NS(C)$, the set of its nonsupport points, is nonempty (this is always the case if X is Banach separable; if X is Fréchet, $NS(C)$ is residual in C). If X is normed, we prove that any locally Lipschitz, convex real function f on C is subdifferentiable on $NS(C)$. If in addition X is Banach separable, we prove that f is smooth on a residual subset of $NS(C)$.

Let X be a topological vector space and X^* be its topological dual. Let C be a closed convex subset of X. Recall that a point x_0 of C is called a support point of C if there exists a nonzero f in X^* such that $f(x) \leq f(x_0)$ for any x in C. We denote by $S(C)$ the set of all support points of C and by $NS(C)$ the set of all nonsupport points of C. If $NS(C)$ is nonempty, then it is convex and has the following property: for any x_0 in $NS(C)$ and x in C, the segment $[x_0, x)$ is contained in $NS(C)$; in particular $NS(C)$ is dense in C. As a matter of fact a stronger result holds if X is Fréchet: $NS(C)$ is a residual subset of C. To see this it is sufficient to show that $S(C)$ is a set of first category in C. This last assertion is an immediate consequence of Theorem 1 in [**5**] (asserting that $S(C)$ is an F_σ) and of the fact that $NS(C)$ is dense in C.

If X is a Banach space, it is well known (Bishop-Phelps theorem, see for example [**3**]) that $S(C)$ is dense in the boundary of C. As a matter of fact, if C has nonempty interior, then $NS(C)$ is exactly the interior of C and $S(C)$ is exactly the boundary of C [**3**, p. 64]. Notice also that if C has empty interior, then $NS(C)$, if nonempty, is neither open nor closed in C (as $NS(C)$ and $S(C)$ are both dense in C). It is obvious that if C is contained in a closed hyperplane, then $NS(C)$ is empty. Even if C is not contained in a closed hyperplane, it may happen that $NS(C)$ is empty [**4**, Example 3, p. 460]. However, if X is separable and C is not contained in a closed hyperplane, then $NS(C)$ is nonempty [**2**, p. 655]. It follows that, under this assumption on X, $NS(C) \neq \emptyset$ iff $\mathrm{cl}(\mathrm{aff}\, C) = X$ (as usual $\mathrm{aff}(C)$ denotes the smallest affine subspace containing C and cl stands for the topological closure).

Let X be a Hausdorff locally convex space and C be a closed convex subset of X such that $NS(C) \neq \emptyset$. Let

$$\mathrm{cone}_x(C) = \{y \in X; x + ty \in C \text{ for some } t > 0\}$$

and

$$T_x(C) = \mathrm{cl}(\mathrm{cone}_x(C)).$$

Notice that $x \in NS(C)$ if and only if $T_x(C) = X$.

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We can now state and prove our main results.

THEOREM 1. *Let C be a closed convex subset of the normed, vector space X and $f\colon C \to R$ be convex and locally Lipschitz on $NS(C)$. Then*
(i) *f is subdifferentiable on $NS(C)$;*
(ii) *for any x in $NS(C)$ the set of subgradients of f at x is weak*-compact.*

PROOF. Let $x \in NS(C)$ and $v \in \operatorname{cone}_x(C)$. Let $t > 0$ be such that $x + tv \in C$. Define $d_{x,v}\colon (0,t) \to R$ by

$$d_{x,v}(h) = \frac{f(x+hv) - f(x)}{h}.$$

$d_{x,v}$ is increasing (as f is convex) and bounded (as f is locally Lipschitz on $NS(C)$). It follows that $\lim_{h\downarrow 0} d_{x,v}(h)$ exists and is finite; denote it $f'(x;v)$. Since f is convex, $f'(x;v)\colon \operatorname{cone}_x(C) \to R$ is sublinear. The fact that f is locally Lipschitz on $NS(C)$ implies that $f'(x;\ )$ is Lipschitz on $\operatorname{cone}_x(C)$. Indeed there exist ε_0, $M > 0$ such that f is Lipschitz on $B(x,\varepsilon_0) \cap NS(C)$ with Lipschitz constant M. Let $v, w \in \operatorname{cone}_x(C)$ and h sufficiently small such that $x + hv$, $x + hw \in B(x,\varepsilon_0) \cap NS(C)$. Then

$$\frac{|f(x+hv) - f(x+hw)|}{h} \le M\|v - w\|$$

and therefore

$$|f'(x;v) - f'(x;w)| = \lim_{h\downarrow 0} \frac{|f(x+hv) - f(x+hw)|}{h} \le M\|v - w\|.$$

As a consequence, there exists a unique continuous $F(x;\ )\colon X \to R$ extending $f'(x;\ )$ (recall that $T_xC = X$). Clearly $F(x;\ )$ is sublinear and continuous (in fact Lipschitz with Lipschitz constant M), $F(x;0) = 0$ and for any $y \in C$, $F(x;y-x) \le f(y) - f(x)$. Assertion (i) follows now from the following remarks: (1) any subgradient of $F(x;\ )$ at zero is a subgradient of f at x; (2) the set of subgradients of $F(x;\ )$ at zero is nonempty. The proof of assertion (ii) is similar to the proof of Theorem B, p. 84 in [**3**].

THEOREM 2. *Let X be a separable Banach space, C be a closed, convex subset of X and $f\colon C \to R$ be convex and locally Lipschitz on $NS(C)$. Then f is smooth on a residual subset of $NS(C)$.*

PROOF. We use the notation introduced above. Recall that for each $x \in NS(C)$ and $v \in X$, $-F(x;-v) \le F(x,v)$. Since X is separable there exists a countable dense subset $(v_n)_n$ of X. For any positive integers m and n let

$$A_{m,n} = \{x \in NS(C); F(x;v_n) + F(x;-v_n) \ge 1/m\}$$

and let $A = \bigcup A_{m,n}$. We shall prove that each $A_{m,n}$ is closed in $NS(C)$ and has empty interior in $NS(C)$. Then for any $x \in NS(C) \setminus A$, $F(x;v) = -F(x;-v)$, which implies that $F(x;\ )$ is linear and therefore is a gradient of f at x.

Let $v \in X$ and let

$$A_{m,v} = \{x \in NS(C); F(x;v) + F(x;-v) \ge 1/m\},$$

m positive integer. First let us prove that each $A_{m,v}$ is closed in $NS(C)$. Assume the contrary: there exists a sequence $(x_n)_n$ in $A_{m,v}$ such that $\lim x_n = x_0 \in NS(C)$, but $x_0 \notin A_{m,v}$; that is, $F(x_0; v) + F(x_0; -v) < 1/m$. Let $\varepsilon > 0$ be such that

$$F(x_0; v) + F(x_0; -v) = 1/m - \varepsilon. \tag{1}$$

As $T_{x_0}(C) = X$, there exist $u', u'' \in \mathrm{cone}_{x_0}(C)$ such that

$$\|v - u'\| < \varepsilon/16M \quad \text{and} \quad \|v + u''\| < \varepsilon/16M$$

(M being the Lipschitz constant of f on $B(x_0, \varepsilon_0)$). We have

$$|F(x_0; v) - F(x_0; u')| \leq M\|v - u'\| < \varepsilon/16$$

and

$$|F(x_0; -v) - F(x_0; u'')| \leq M\|v + u''\| < \varepsilon/16.$$

It follows that $F(x_0; u') + F(x_0; u'') < 1/m - 7\varepsilon/8$; that is

$$f'(x_0; u') + f'(x_0; u'') < 1/m - 7\varepsilon/8$$

and therefore, for some $t > 0$, we have

$$\frac{f(x_0 + tu') - f(x_0)}{t} + \frac{f(x_0 + tu'') - f(x_0)}{t} < \frac{1}{m} - \frac{7}{8}\varepsilon. \tag{2}$$

Let $v'_n = u' + (x_0 - x_n)/t$ and $v''_n = u'' + (x_0 - x_n)/t$, $n \in N$. Then $x_0 + tu' = x_n + tv'_n$ and $x_0 + tu'' = x_n + tv''_n$, implying that $v'_n, v''_n \in \mathrm{cone}_{x_n}(C)$. Now (2) becomes

$$\frac{f(x_n + tv'_n) - f(x_n)}{t} + \frac{f(x_n + tv''_n) - f(x_n)}{t} + \frac{2(f(x_n) - f(x_0))}{t} \leq \frac{1}{m} - \frac{7}{8}\varepsilon,$$

hence

$$f'(x_n; v'_n) + f'(x_n; v''_n) < \frac{1}{m} - \frac{7}{8}\varepsilon - \frac{2(f(x_n) - f(x_0))}{t}.$$

For large n, $|f(x_n) - f(x_0)| < \varepsilon t/16$ (since $\lim x_n = x_0$); so

$$F(x_n; v'_n) + F(x_n; v''_n) < 1/m - 3\varepsilon/4. \tag{3}$$

Since $x_n \in A_{m,v}$, one has $F(x_n; v) + F(x_n; -v) \geq 1/m$, $n \in N$. For large n, $x_n \in B(x_0, \varepsilon_0) \cap NS(C)$ and therefore

$$\begin{aligned}\|v - v'_n\| &\leq \|v - u'\| + \|u' - v'_n\| < \frac{\varepsilon}{16M} + \frac{\|x_n - x_0\|}{t} \\ &< \frac{\varepsilon}{16M} + \frac{\varepsilon}{16M} = \frac{\varepsilon}{8M}\end{aligned}$$

(take n large enough such that $\|x_n - x_0\| < t\varepsilon/16M$). Similarly

$$\|v + v''_n\| \leq \|v + u''\| + \|v''_n - u''\| < \varepsilon/8M.$$

At this point it is important to observe that $F(x;\ )$ is Lipschitz on X with the same Lipschitz constant M for all $x \in B(x_0, \varepsilon_0) \cap NS(C)$, hence

$$|F(x_n; v) - F(x_n; v'_n)| \leq M\|v - v'_n\| < \varepsilon/8$$

and

$$|F(x_n; -v) - F(x_n; v''_n)| \leq M\|v + v''_n\| < \varepsilon/8.$$

Consequently

$$F(x_n; v_n') + F(x_n; v_n'') > F(x_n; v) + F(x_n; -v) - \varepsilon/4$$

for large n. As $x_n \in A_{m,v}$ this is in conflict with (3), proving that $x_0 \in A_{m,v}$.

Assume now that the interior of $A_{m,v}$ in $NS(C)$ is nonempty. Let $x_0 \in \operatorname{int}_{NS(C)}(A_{m,v})$; then $B(x_0, \varepsilon) \cap NS(C) \subset A_{m,v}$ for some $\varepsilon > 0$. For $x \in B(x_0, \varepsilon) \cap NS(C)$, $F(x; v) + F(x; -v) \geq 1/m$. As $T_{x_0}(C) = X$, there exists $v' \in \operatorname{cone}_{x_0}(C)$ such that $\|v - v'\| < 1/2mM$ and $x_0 + t_0 v' \in C$ for some $t_0 > 0$. We have $x_0 + tv' \in B(x_0, \varepsilon) \cap NS(C)$ for $t < t_0$ sufficiently small; but then $[x_0, x_0 + tv'] \subset B(x_0, \varepsilon) \cap NS(C)$. Define $g\colon (0, t) \to R$ by $g(s) = f(x_0 + sv')$. Since g is convex, there exists $s_0 \in (0, t)$ such that $g'_+(s_0) = g'_-(s_0)$ (g'_+, g'_- denoting as usual the right and left derivatives of g at s_0). But

$$g'_+(s_0) = f'(x_0 + s_0 v'; v')$$

and

$$g'_-(s_0) = -f'(x_0 + s_0 v'; -v').$$

Thus $f'(x_0 + s_0 v'; v') = -f'(x_0 + s_0 v'; -v')$, or $F(x_0 + s_0 v'; v') + F(x_0 + s_0 v'; -v') = 0$ contradicting the fact that $x_0 + s_0 v' \in A_{m,v}$. Thus, the interior of $A_{m,v}$ in $NS(C)$ is empty.

COROLLARY. *Let $f\colon C \to R$ be convex and locally Lipschitz on $NS(C)$. Then the set of those x in $NS(C)$ at which f fails to be smooth cannot contain any closed convex subset K such that* $\operatorname{cl}(\operatorname{aff} K) = X$.

REMARK. If the interior of C is nonempty, the same conclusions hold, but with the apparently weaker assumption that f is lower semicontinuous on $\operatorname{int}(C)$ (see for example [**2**, p. 167]). However, in this case the semicontinuity of f is equivalent with f being locally Lipschitz on $\operatorname{int}(C)$. Without any assumption on the interior of C, it is known that the lower semicontinuity of f implies its subdifferentiability on a dense subset of C (if X is Banach; see [**1**]). In our case, by imposing stronger conditions on f we obtain its subdifferentiability on the whole $NS(C)$ and its smoothness on a residual subset of $NS(C)$, and hence of C.

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