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APPROXIMATION BY OPERATORS WITH FIXED NULLITY

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ABSTRACT. Let T be a fixed operator on a complex separable Hilbert space H. The distance from T to the operators with nullity equal to n, for each possible value of n, is determined.

1. Introduction. Let H be a fixed complex separable Hilbert space. For any (bounded linear) operator T on H we define the nullity and deficiency, denoted $\operatorname{nul} T$ and $\operatorname{def} T$, to be the dimensions of the kernels of T and T^*, respectively. Of course, the index of T, denoted $\operatorname{ind} T$, is defined to be $(\operatorname{nul} T - \operatorname{def} T)$, with $\infty - \infty$ understood to be 0. We denote the operator norm of T by $\|T\|$.

In [**4**] the distance from an arbitrary operator to the invertible operators (and to the Fredholm operators) was determined. This refined some results in [**6**] that described the closure of the invertible operators. The results and methods of [**4**] found application in [**2**, **9**, **12**], and other research. In this note we continue this sequence of results by determining the distance from a fixed operator T to the set of operators with a given fixed nullity. The results complement [**2** and **12**]. The reader interested in operator approximations is also directed to [**3**, **7**, **8**], and the bibliography of [**2**].

Recall that the reduced minimum modulus of T, denoted $\gamma(T)$, is defined by $\gamma(T) = \inf\{\|Tf\|: \|f\| = 1, f \perp \ker T\}$. It is well known that the range of T, denoted TH, is closed if and only if $\gamma(T) > 0$. The essential spectrum of an operator T, denoted $\sigma_e(T)$, is the set $\{z: T - zI \text{ is not a Fredholm operator}\}$. We define the essential minimum modulus $m_e(T)$ by $m_e(T) = \inf\{\lambda: \lambda \in \sigma_e((T^*T)^{1/2})\}$. The following enumeration of properties of $m_e(T)$, was developed in [**4**]. (i) If $E(\cdot)$ is the spectral measure for $R = (T^*T)^{1/2}$ then the smallest nonnegative α such that $E([\alpha, \alpha+\delta))H$ is infinite dimensional for every positive δ is $\alpha = m_e(T)$. (ii) The range TH is closed and $\operatorname{nul} T$ is finite if and only if $m_e(T) > 0$. (iii) The range T^*H is closed and $\operatorname{def} T$ is finite if and only if $m_e(T^*) > 0$. (iv) The operator T is Fredholm if and only if $m_e(T)$ and $m_e(T^*)$ are positive. In that case, $m_e(T) = m_e(T^*)$.

2. Main results. To simplify subsequent notation we make the following definition.

DEFINITION. For n any nonnegative integer or ∞ we define $\rho_n(T)$ by

$$\rho_n(T) = \inf\{\|T - A\|: \operatorname{nul} A = n\}.$$

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This brings us to our first theorem. The formula given here for the distance $\rho_n(T)$ is unlike those appearing in [**4**, **2**, **11**, and **12**]. It gives a significant refinement and sharpening of Proposition 4.7 in [**12**].

THEOREM 1. *Assume that* $n > \operatorname{nul} T$.

(i) *We always have* $m_e(T) \geq \rho_n(T) \geq \gamma(T)$. *Consequently, if* $m_e(T) = \gamma(T)$ *then this common value is* $\rho_n(T)$.

(ii) *If* $m_e(T) > \gamma(T)$ *then* $\rho_n(T) = \sup\{\lambda\colon \dim E([0,\lambda))H < n\}$ *where* UR *is the usual polar factorization of* T *and* $E(\cdot)$ *is the spectral measure for* R.

(iii) *For* $n \geq 2$ *and* $m_e(T) > 0$, $\rho_n(T)$ *can assume any value in the interval* $[\gamma(T), m_e(T)]$.

PROOF OF (i). For A such that $\operatorname{nul} A = n > \operatorname{nul} T$ it is immediate from the properties of $\gamma(T)$ (see [**5**, pp. 364–365]) that $\|T - A\| \geq \gamma(T)$. Hence, $\rho_n(T) \geq \gamma(T)$ and it suffices to show that $m_e(T) \geq \rho_n(T)$.

Let UR be the polar factorization of T with $\ker U = \ker R$. It is routine to verify that $\ker T = \ker R$ and $m_e(T) = m_e(R)$. Using property (i) of the essential minimum modulus for R we construct an orthonormal sequence $\{f_1, f_2, \dots\}$ such that $\{Rf_1, Rf_2, \dots\}$ is an orthogonal set and $\{\|Rf_1\|, \|Rf_2\|, \dots\}$ is a nonincreasing sequence converging to $m_e(R)$. One possible construction begins by letting $I_m = (m_e(R) + 2^{-m-1}, m_e(R) + 2^{-m}]$ for $m = 1, 2, \dots$ and noting that $\{E(I_1), E(I_2), \dots\}$ is pairwise orthogonal where $E(\cdot)$ is the spectral measure for R. To avoid contradicting property (i) we conclude that either an infinite number of the projections $\{E(I_1), E(I_2), \dots\}$ are nonzero or else $m_e(R)$ is an infinite dimensional eigenvalue for R. In the first case, pick a unit vector $f_k \in E(I_k)H$ if $E(I_k) \neq 0$ and in the second, simply choose an orthonormal basis for the eigenspace corresponding to $m_e(R)$. Note that in the latter case it follows that $m_e(R) > 0$ for otherwise $\operatorname{nul} T = \operatorname{nul} R = \infty$, contradicting that $n > \operatorname{nul} T$.

Let P_m be the orthogonal projection onto $\{f_m, \dots, f_{m+p}\}$ where $p+1 = n - \operatorname{nul} T$ and define A_m by $A_m = T(I - P_m)$. Note that $\ker A_m = \ker T \oplus P_m H$ and $\operatorname{nul} A_m = \operatorname{nul} T + (p+1) = n$. For f in H we have

$$\begin{aligned} \|(T - A_m)f\|^2 = \|RP_m f\|^2 &= \sum_{j=0}^{p} |c_j|^2 \|Rf_{m+j}\|^2 \\ &\leq \|Rf_m\|^2 \sum_{j=0}^{p} |c_j|^2 = \|Rf_m\|^2 \|P_m f\|^2. \end{aligned}$$

Thus,

$$\|T - A_m\| \leq \|Rf_m\| \quad \text{and} \quad \rho_n(T) \leq \inf_m \|T - A_m\| \leq \inf_m \|Rf_m\| = m_e(T),$$

which completes the proof of (i).

PROOF OF (ii). Since $m_e(T)$ is positive, property (ii) of the essential minimum modulus implies that the range of T is closed, which implies that $\gamma(T) > 0$.

Choose A and λ such that $\operatorname{nul} A = n$ and $\dim E([0,\lambda))H < n$; let P denote the projection $E([0,\lambda))$. Note that

$$\|T - A\| \geq \|(T - A)|\ker A\| \geq \|(I - P)R|\ker A\| = \|R|(I - P)\ker A\| \geq \lambda$$

provided that $(I-P)\ker A$ is nontrivial. Clearly $\operatorname{nul}(I-P) < n = \operatorname{nul} A$ and thus $(I-P)\ker A$ must be nontrivial. From the inequality deduced above it follows that $\rho_n(T) \geq \lambda$ and

$$(*) \qquad \rho_n(T) \geq \mu \equiv \sup\{\lambda\colon \dim E([0,\lambda))H < n\}.$$

Now we prove the inequality opposite to $(*)$. From the usual properties of the spectral measure it follows that $\dim E([0,\mu+1/k))H \geq n$. Choose M_k to be a subspace of $E([0,\mu+1/k))H$ that contains $E([0,\mu))H$ and such that $\dim M_k = n$. Let P_k be the orthogonal projection onto M_k and note that $\operatorname{nul} T(I-P_k) = \dim M_k = n$. If A_k is $T(I-P_k)$ then $\|T-A_k\| \leq \mu + 1/k$ and $\rho_n(T) \leq \inf \|T-A_k\| \leq \mu$, as desired.

PROOF OF (iii). If $\gamma(T) = m_e(T)$ then there is nothing to prove. Choose two positive numbers a and c such that $a < c$; choose b such that $a \leq b \leq c$. Choose n to be some integer such that $n \geq 2$. We shall construct T such that $\gamma(T) = a$, $\rho_n(T) = b$, and $m_e(T) = c$. Of course, $m_e(T) > 0$ implies that the range of T is closed, which implies that $\gamma(T) > 0$. Define T to be the diagonal operator with the first $n-1$ entries equal to a, the next two entries equal to b and all of the remaining entries equal to c. Note that $T = R$. Let D be the diagonal operator with the first n entries equal to 0, the $n+1$ entry equal to b, and all other entries equal to c. Clearly we have $\|T-D\| = b$ and $\ker D = n$; thus, $\rho_n(T) \leq b$.

Let P be the orthogonal projection onto $\ker(T-a)$ and let A be any operator with $\operatorname{nul} A = n$. Note that

$$\|T-A\| \geq \|(I-P)R|\ker A\| = \|R|(I-P)\ker A\| \geq b;$$

the last inequality follows because $(I-P)\ker A$ must be nontrivial. It follows that $\rho_n(T) \geq b$; this inequality and the preceding paragraph show that $\rho_n(T) = b$, as desired. It is routine to verify that $\gamma(T) = a$ and $m_e(T) = c$.

In the next theorem the formula for $\rho_n(T)$ resembles previously obtained distance formulas. This theorem provides a notable sharpening and simplification of Proposition 3.10 in [**12**]. It gives an interesting contrast to Theorem 12.2 of [**2**, p. 146].

THEOREM 2. *Assume that* $n < \operatorname{nul} T$.
(i) *If* $n \geq \operatorname{ind} T$ *then* $\rho_n(T) = 0$.
(ii) *If* $n < \operatorname{ind} T$ *then* $\rho_n(T) = m_e(T^*)$.

PROOF OF (i). Note that $\operatorname{def} T \geq \operatorname{nul} T - n$. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for $\ker T$ and note that this basis could be finite or infinite. Let $\{f_1, f_2, \ldots\}$ be an orthonormal basis for $\ker T^*$ and note that it is infinite if the basis for $\ker T$ is infinite. Define A_k by

$$\begin{aligned} A_k e_j &= 0, \qquad j = 1, 2, \ldots, n, \\ A_k e_{n+j} &= (1/k) f_j, \qquad j = 1, 2, \ldots, \\ A_k|(\ker T)^\perp &= T|(\ker T)^\perp. \end{aligned}$$

It is routine to verify that $\operatorname{nul} A_k = n$ and $\|T - A_k\| = 1/k$. This proves (i).

PROOF OF (ii). This part of the proof requires the consideration of cases. Using the methods of [**4**] to refine Theorem 12.2 in [**2**, p. 146], Wu shows in [**12**, Theorem 3.1] that $\rho_n(T) \leq \max\{m_e(T), m_e(T^*)\}$. A folklore result, which is proved in [**11**], indicates that $\operatorname{ind} A = \operatorname{ind} T$ whenever $\|T - A\| < m_e(T)$. This result implies that $\rho_n(T) \geq \max\{m_e(T), m_e(T^*)\}$. Thus, we have $\rho_n(T) = \max\{m_e(T), m_e(T^*)\}$ and we need only argue that the right side of the preceding equation equals $m_e(T^*)$. Property (iv) of the essential minimum modulus gives the desired conclusion whenever $m_e(T^*)$ is positive. Because $\operatorname{ind} T$ exceeds n, which is a nonnegative integer, we conclude that $\operatorname{nul} T > \operatorname{def} T$. Thus, $\operatorname{def} T$ is finite and $m_e(T^*) = 0$ implies that the range of T^* is not closed according to property (iii) of the essential minimum modulus. Thus, the range of T is not closed and $m_e(T) = 0$ according to property (ii). This proves that

$$\max\{m_e(T), m_e(T^*)\} = 0 = m_e(T^*)$$

and concludes the proof of (ii).

By taking adjoints and using various elementary results we can restate Theorems 1 and 2, respectively, as formulas for approximation by operators with fixed deficiency.

COROLLARY. *Let* $\eta_n(T) = \inf\{\|T-A\|: \operatorname{def} A = n\}$ *and assume that* $n > \operatorname{def} T$.

(i) $m_e(T^*) \geq \eta_n(T) \geq \gamma(T)$.

(ii) *If* $m_e(T^*) > \gamma(T)$ *then* $\eta_n(T) = \sup\{\lambda\colon \dim E([0, \lambda))H < n\}$ *where* $E(\cdot)$ *is the spectral measure for* $(TT^*)^{1/2}$.

(iii) *For* $n \geq 2$ *and* $m_e(T^*) > 0$, $\eta_n(T)$ *can assume any value in the interval* $[\gamma(T), m_e(T^*)]$.

COROLLARY. *Assume that* $n < \operatorname{def} T$.

(i) *If* $n \geq -\operatorname{ind} T$ *then* $\eta_n(T) = 0$.

(ii) *If* $n < -\operatorname{ind} T$ *then* $\eta_n(T) = m_e(T)$.

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