APPROXIMATION BY OPERATORS WITH FIXED NULLITY

RICHARD BOULDIN

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ABSTRACT. Let $T$ be a fixed operator on a complex separable Hilbert space $H$. The distance from $T$ to the operators with nullity equal to $n$, for each possible value of $n$, is determined.

1. Introduction. Let $H$ be a fixed complex separable Hilbert space. For any (bounded linear) operator $T$ on $H$ we define the nullity and deficiency, denoted nul $T$ and def $T$, to be the dimensions of the kernels of $T$ and $T^*$, respectively. Of course, the index of $T$, denoted ind $T$, is defined to be $(\text{nul} T - \text{def} T)$, with $\infty - \infty$ understood to be $0$. We denote the operator norm of $T$ by $\|T\|$.

In [4] the distance from an arbitrary operator to the invertible operators (and to the Fredholm operators) was determined. This refined some results in [6] that described the closure of the invertible operators. The results and methods of [4] found application in [2, 9, 12], and other research. In this note we continue this sequence of results by determining the distance from a fixed operator $T$ to the set of operators with a given fixed nullity. The results complement [2 and 12]. The reader interested in operator approximations is also directed to [3, 7, 8], and the bibliography of [2].

Recall that the reduced minimum modulus of $T$, denoted $\gamma(T)$, is defined by $\gamma(T) = \inf\{\|Tf\| : \|f\| = 1, f \perp \ker T\}$. It is well known that the range of $T$, denoted $TH$, is closed if and only if $\gamma(T) > 0$. The essential spectrum of an operator $T$, denoted $\sigma_e(T)$, is the set $\{z : T - zI$ is not a Fredholm operator\}. We define the essential minimum modulus $m_e(T)$ by $m_e(T) = \inf\{\lambda : \lambda \in \sigma_e((T^*T)^{1/2})\}$. The following enumeration of properties of $m_e(T)$, was developed in [4]. (i) If $E(\cdot)$ is the spectral measure for $R = (T^*T)^{1/2}$ then the smallest nonnegative $\alpha$ such that $E([\alpha, \alpha + \delta))H$ is infinite dimensional for every positive $\delta$ is $\alpha = m_e(T)$. (ii) The range $TH$ is closed and nul $T$ is finite if and only if $m_e(T) > 0$. (iii) The range $T^*H$ is closed and def $T$ is finite if and only if $m_e(T^*) > 0$. (iv) The operator $T$ is Fredholm if and only if $m_e(T)$ and $m_e(T^*)$ are positive. In that case, $m_e(T) = m_e(T^*)$.

2. Main results. To simplify subsequent notation we make the following definition.

DEFINITION. For $n$ any nonnegative integer or $\infty$ we define $\rho_n(T)$ by

$$\rho_n(T) = \inf\{\|T - A\| : \text{nul} A = n\}.$$

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This brings us to our first theorem. The formula given here for the distance $p_n(T)$ is unlike those appearing in [4, 2, 11, and 12]. It gives a significant refinement and sharpening of Proposition 4.7 in [12].

**THEOREM 1.** Assume that $n > \text{nul } T$.

(i) We always have $m_e(T) \geq \gamma(T) \geq \rho_n(T)$. Consequently, if $m_e(T) = \gamma(T)$ then this common value is $\rho_n(T)$.

(ii) If $m_e(T) > \gamma(T)$ then $\rho_n(T) = \sup \{ \lambda : \text{dim } E([0, \lambda)) H < n \}$ where UR is the usual polar factorization of $T$ and $E(\cdot)$ is the spectral measure for $R$.

(iii) For $n \geq 2$ and $m_e(T) > 0$, $\rho_n(T)$ can assume any value in the interval $[\gamma(T), m_e(T)]$.

**PROOF OF (i).** For $A$ such that $\text{nul } A = n > \text{nul } T$ it is immediate from the properties of $\gamma(T)$ (see [5, pp. 364-365]) that $\|T - A\| \geq \gamma(T)$. Hence, $\rho_n(T) \geq \gamma(T)$ and it suffices to show that $m_e(T) \geq \rho_n(T)$.

Let $UR$ be the polar factorization of $T$ with ker $U = \text{ker } R$. It is routine to verify that ker $T = \text{ker } R$ and $m_e(T) = m_e(R)$. Using property (i) of the essential minimum modulus for $R$ we construct an orthonormal sequence $\{f_1, f_2, \ldots\}$ such that $\{Rf_1, Rf_2, \ldots\}$ is an orthogonal set and $\{\|Rf_1\|, \|Rf_2\|, \ldots\}$ is a non-increasing sequence converging to $m_e(R)$. One possible construction begins by letting $I_m = (m_e(R) + 2^{-m-1}, m_e(R) + 2^{-m}]$ for $m = 1, 2, \ldots$ and noting that $\{E(I_1), E(I_2), \ldots\}$ is pairwise orthogonal where $E(\cdot)$ is the spectral measure for $R$. To avoid contradicting property (i) we conclude that either an infinite number of the projections $\{E(I_1), E(I_2), \ldots\}$ are nonzero or else $m_e(R)$ is an infinite dimensional eigenvalue for $R$. In the first case, pick a unit vector $f_k \in E(I_k)H$ if $E(I_k) \neq 0$ and in the second, simply choose an orthonormal basis for the eigenspace corresponding to $m_e(R)$. Note that in the latter case it follows that $m_e(R) > 0$ for otherwise $\text{nul } T = \text{nul } R = \infty$, contradicting that $n > \text{nul } T$.

Let $P_m$ be the orthogonal projection onto $\{f_m, \ldots, f_{m+p}\}$ where $p+1 = n - \text{nul } T$ and define $A_m$ by $A_m = T(I - P_m)$. Note that ker $A_m = \text{ker } T \oplus P_mH$ and $\text{nul } A_m = \text{nul } T + (p + 1) = n$. For $f$ in $H$ we have

$$\|T - A_m\| = \|RP_m f\|^2 = \sum_{j=0}^{p} |c_j|^2 \|Rf_{m+j}\|^2 \leq \|Rf_m\|^2 \sum_{j=0}^{p} |c_j|^2 = \|Rf_m\|^2 \|P_m f\|^2.$$ 

Thus,

$$\|T - A_m\| \leq \|Rf_m\| \quad \text{and} \quad \rho_n(T) \leq \inf_m \|T - A_m\| \leq \inf_m \|Rf_m\| = m_e(T),$$

which completes the proof of (i).

**PROOF OF (ii).** Since $m_e(T)$ is positive, property (ii) of the essential minimum modulus implies that the range of $T$ is closed, which implies that $\gamma(T) > 0$.

Choose $A$ and $\lambda$ such that $\text{nul } A = n$ and dim $E([0, \lambda)) H < n$; let $P$ denote the projection $E([0, \lambda))$. Note that

$$\|T - A\| \geq \|(T - A) \text{ ker } A\| \geq \|(I - P)R \text{ ker } A\| = \|R(I - P) \text{ ker } A\| \geq \lambda$$
provided that \((I - P)\ker A\) is nontrivial. Clearly \(\text{null}(I - P) < n = \text{null} A\) and thus \((I - P)\ker A\) must be nontrivial. From the inequality deduced above it follows that 
\[
\rho_n(T) \geq \lambda \equiv \sup\{\lambda : \dim E([0, \lambda])H < n\}.
\]

\((*)\)

Now we prove the inequality opposite to \((*)\). From the usual properties of the spectral measure it follows that \(\dim E([0, \mu + 1/k])H \geq n\). Choose \(M_k\) to be a subspace of \(E([0, \mu + 1/k])H\) that contains \(E([0, \mu])H\) and such that \(\dim M_k = n\). Let \(P_k\) be the orthogonal projection onto \(M_k\) and note that \(\text{null} T(I - P_k) = \dim M_k = n\). If \(A_k\) is \(T(I - P_k)\) then \(\|T - A_k\| \leq \mu + 1/k\) and \(\rho_n(T) \leq \inf \|T - A_k\| \leq \mu\), as desired.

**Proof of (iii).** If \(\gamma(T) = m_e(T)\) then there is nothing to prove. Choose two positive numbers \(a\) and \(c\) such that \(a < c\); choose \(b\) such that \(a \leq b \leq c\). Choose \(n\) to be some integer such that \(n \geq 2\). We shall construct \(T\) such that \(\gamma(T) = a\), \(\rho_n(T) = b\), and \(m_e(T) = c\). Of course, \(m_e(T) > 0\) implies that the range of \(T\) is closed, which implies that \(\gamma(T) > 0\). Define \(T\) to be the diagonal operator with the first \(n - 1\) entries equal to \(a\), the next two entries equal to \(b\) and all of the remaining entries equal to \(c\). Note that \(T = R\). Let \(D\) be the diagonal operator with the first \(n\) entries equal to \(0\), the \(n + 1\) entry equal to \(b\), and all other entries equal to \(c\). Clearly we have \(\|T - D\| = b\) and \(\ker D = n\); thus, \(\rho_n(T) \leq b\).

Let \(P\) be the orthogonal projection onto \(\ker(T - a)\) and let \(A\) be any operator with \(\text{null} A = n\). Note that

\[
\|T - A\| \geq \|(I - P)R\| \ker A\| = \|R\|(I - P)\ker A\| \geq b;
\]
the last inequality follows because \((I - P)\ker A\) must be nontrivial. It follows that \(\rho_n(T) \geq b\); this inequality and the preceding paragraph show that \(\rho_n(T) = b\), as desired. It is routine to verify that \(\gamma(T) = a\) and \(m_e(T) = c\).

In the next theorem the formula for \(\rho_n(T)\) resembles previously obtained distance formulas. This theorem provides a notable sharpening and simplification of Proposition 3.10 in [12]. It gives an interesting contrast to Theorem 12.2 of [2, p. 146].

**Theorem 2.** Assume that \(n < \text{null} T\).

(i) If \(n \geq \text{ind} T\) then \(\rho_n(T) = 0\).

(ii) If \(n < \text{ind} T\) then \(\rho_n(T) = m_e(T^*)\).

**Proof of (i).** Note that \(\text{def} T \geq \text{null} T - n\). Let \(\{e_1, e_2, \ldots\}\) be an orthonormal basis for \(\ker T\) and note that this basis could be finite or infinite. Let \(\{f_1, f_2, \ldots\}\) be an orthonormal basis for \(\ker T^*\) and note that it is infinite if the basis for \(\ker T\) is infinite. Define \(A_k\) by

\[
A_k e_j = 0, \quad j = 1, 2, \ldots, n,
\]

\[
A_k e_{n+j} = (1/k)f_j, \quad j = 1, 2, \ldots,
\]

\[
A_k|(\ker T)^\perp = T|(\ker T)^\perp.
\]

It is routine to verify that \(\text{null} A_k = n\) and \(\|T - A_k\| = 1/k\). This proves (i).

**Proof of (ii).** This part of the proof requires the consideration of cases. Using the methods of [4] to refine Theorem 12.2 in [2, p. 146], Wu shows in [12, Theorem 3.1] that \(\rho_n(T) \leq \max\{m_e(T), m_e(T^*)\}\). A folklore result, which is proved in [11],
indicates that \( \text{ind} A = \text{ind} T \) whenever \( \| T - A \| < m_e(T) \). This result implies that 
\[
\rho_n(T) \geq \max\{m_e(T), m_e(T^*)\}. 
\]
Thus, we have \( \rho_n(T) = \max\{m_e(T), m_e(T^*)\} \) and we need only argue that the right side of the preceding equation equals \( m_e(T^*) \). Property (iv) of the essential minimum modulus gives the desired conclusion whenever \( m_e(T^*) \) is positive. Because \( \text{ind} T \) exceeds \( n \), which is a nonnegative integer, we conclude that \( \text{nul} T > \text{def} T \). Thus, \( \text{def} T \) is finite and \( m_e(T^*) = 0 \) implies that the range of \( T^* \) is not closed according to property (iii) of the essential minimum modulus. Thus, the range of \( T \) is not closed and \( m_e(T) = 0 \) according to property (ii). This proves that 
\[
\max\{m_e(T), m_e(T^*)\} = 0 = m_e(T^*) 
\]
and concludes the proof of (ii).

By taking adjoints and using various elementary results we can restate Theorems 1 and 2, respectively, as formulas for approximation by operators with fixed deficiency.

**COROLLARY.** Let \( \eta_n(T) = \inf\{\| T - A \| : \text{def} A = n \} \) and assume that \( n > \text{def} T \).

(i) \( m_e(T^*) \geq \eta_n(T) \geq \gamma(T) \).

(ii) If \( m_e(T^*) > \gamma(T) \) then \( \eta_n(T) = \sup\{\lambda : \dim E([0, \lambda)) \subset \text{def} T^* \} \).

(iii) For \( n \geq 2 \) and \( m_e(T^*) > 0 \), \( \eta_n(T) \) can assume any value in the interval \([\gamma(T), m_e(T^*)]\).

**COROLLARY.** Assume that \( n < \text{def} T \).

(i) If \( n \geq -\text{ind} T \) then \( \eta_n(T) = 0 \).

(ii) If \( n < -\text{ind} T \) then \( \eta_n(T) = m_e(T) \).

**REFERENCES**


Department of Mathematics, University of Georgia, Athens, Georgia 30602