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A CHARACTERIZATION OF NONLINEAR SEMIGROUPS WITH SMOOTHING EFFECT

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ABSTRACT. Let $\{S(t) \colon t > 0\}$ be a nonlinear semigroup of operators mapping a closed subset C of a real Banach space X into itself. Conditions are found for an accretive operator in X to be the generator of $\{S(t) \colon t > 0\}$ with smoothing effect:

$$\text{For each } x \in C, \ S(t)x \in V \text{ a.e. } t > 0,$$

among other things, where V is a Banach space imbedded continuously in X.

The conditions contain a Gårding-type inequality, and are shown also to be necessary if C is a closed convex subset of a "nice" Banach space X.

1. Introduction. Let $(X, |\cdot|)$ and $(V, \|\cdot\|)$ be real Banach spaces and assume that V is imbedded continuously in X. In this paper, we are concerned with the semigroup $\{S(t) \colon t > 0\}$ of operators mapping the closed subset C of X into itself, which satisfies the following conditions:

(S.1) For each $x \in C$ and $t > 0$, $S(\cdot)x$ belongs to $C([0,\infty); X) \cap L^p(0,t;V)$ and satisfies

$$|S(t)x - S(t)y|^p + K \int_0^t \|S(r)x - S(r)y\|^p \, dr \leq |x-y|^p \tag{1}$$

for $x, y \in C$, where $p > 1$ and $K > 0$ are constants.

(S.2) Any $u \in C$ such that $|t^{-1}(u - S(t)u)|$ is bounded for $0 < t < 1$, belongs to V and $S(\cdot)u$ to $C([0,\infty); V)$.

The author [**11**] recently proved that if an operator A in X is m-accretive and satisfies

$$D(A) \subset V \tag{2}$$

and, for $u, v \in D(A)$ and $\lambda > 0$,

$$|u-v|^p + K\lambda \|u-v\|^p \leq |u + \lambda Au - v - \lambda Av|^p,$$

then the semigroup $\{S(t) \colon t > 0\}$ generated by A in the sense of Crandall and Liggett [**4**, Theorem I]:

$$S(t)x = \lim_{\lambda \downarrow 0} (I + \lambda A)^{-[t/\lambda]} x \quad \text{for } x \in \overline{D(A)} \tag{3}$$

satisfies the condition (S.1) with $C = \overline{D(A)}$, the closure in X of $D(A)$.

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In this paper we shall first find conditions for an accretive operator A in X to generate a semigroup with both properties (S.1) and (S.2); next we shall show that our conditions on A are equivalent to the conditions (S.1) and (S.2), provided that X is a "nice" Banach space such as a Hilbert space or a Banach space with a uniformly convex dual, and that C is a closed convex subset of X.

Suggested by the previous conditions, we consider an accretive subset A of $X \times X$ satisfying the conditions:

(A.1) For $\lambda > 0$, $R(I + \lambda A) \supset C$ with $C = \overline{D(A)}$.

(A.2) Any $u \in C$, such that $|\lambda^{-1}(u - (I + \lambda A)^{-1}u)|$ is bounded for $0 < \lambda < 1$, belongs to V and

$$|u - v|^p + K\lambda\|u - v\|^p \leq |u + \lambda Au - v - \lambda Av|^p \tag{4}$$

holds for $u, v \in D(A)$ and $\lambda > 0$,

where, and in what follows, we abuse the notation Au to denote an element of it if there is no fear of confusion.

According to Crandall [**3**], the first statement of (A.2) may be replaced with $\hat{D}(A) \subset V$, which is generally stronger than (2) since the generalized domain $\hat{D}(A)$ of A includes $D(A)$. The inequality (4) in (A.2) is equivalent to

$$p|u - v|^{p-1}\tau(u - v, Au - Av) \geq K\|u - v\|^p, \tag{5}$$

where $\tau(x, y) = \lim_{\lambda\downarrow 0} \lambda^{-1}(|x + \lambda y| - |x|)$ for $x, y \in X$. Hence it turns out to be a Gårding-type inequality if X is a Hilbert space and $p = 2$. Actually, we pointed out in the previous paper [**11**, Example] that the semilinear operator $-\Delta + (1 + |\operatorname{grad} \cdot|^2)^{1/2}$ with Dirichlet boundary condition satisfies (2) and (5) in the L^2-space setting. In §2 we will give more familiar examples of accretive operators with properties (A.1) and (A.2).

The condition (S.1) implies the "smoothing effect" of the semigroup $\{S(t) : t > 0\}$: For any $x \in C$, $S(t)x$ "belongs to the smaller Banach space" V a.e. $t > 0$. This kind of phenomenon has been studied for the semigroups mainly associated with concrete diffusion equations. Indeed, sharp results have been obtained "case by case" from the particular properties of each given nonlinear diffusion equation (see, e.g. [**2, 5, 9**]). As powerful tools we have the subdifferential of a lower semicontinuous function, and the Lyapunov function for an accretive operator developed in [**9**]. Our intention in the implication

$$\text{(A.1) and (A.2)} \to \text{(S.1) and (S.2)}$$

is to provide another type of method available to this kind of study for a class of diffusion equations rather than to obtain specific results in this direction.

In §3, making additional assumptions on C and X as mentioned above, we will deal with the converse problem:

$$\text{(S.1) and (S.2)} \to \text{(A.1) and (A.2)}.$$

The author considers it to be meaningful from an abstract point of view to characterize the generator of a nonlinear semigroup with properties (S.1) and (S.2). In fact, at present, we have no complete nonlinear analogue of linear differentiable [**8**] or analytic semigroups (cf. [**2**, p. 141]).

2. Conditions for generation. In this section we shall prove the following theorem on the generation part.

THEOREM 1. *Let A be an accretive operator in X with properties* (A.1) *and* (A.2). *Then the semigroup $\{S(t): t > 0\}$ generated by A, through* (3), *satisfies the conditions* (S.1) *and* (S.2) *with $C = \overline{D(A)}$.*

PROOF. Recalling the remark given just after the statements of (A.1) and (A.2), we find that A satisfies the conditions (A.1), (2), and (4). Therefore, using a quite similar method to that used in the proof of [**11**, Theorem 1], we can obtain easily that the semigroup satisfies (S.1). We have thus only to show (S.2) for the semigroup. To this end we need the following:

LEMMA 1. *Let $\{S(t): t > 0\}$ be the semigroup generated by A, through* (3), *under the condition* (A.1). *Then, for each $u \in \overline{D(A)}$, $|t^{-1}(u - S(t)u)|$ is bounded for $0 < t < 1$ if and only if $|\lambda^{-1}(u - (I + \lambda A)^{-1}u)|$ is bounded for $0 < \lambda < 1$.*

The proof of this lemma might be omitted since a more strict assertion has been presented in [**3**, Theorem 1].

Let us turn now to the proof of Theorem 1. Take $u \in \overline{D(A)}$ such that $|t^{-1}(u - S(t)u)|$ is bounded for $0 < t < 1$. Then, by Lemma 1, $|\lambda^{-1}(u - I_\lambda u)|$ is bounded for $0 < \lambda < 1$, where $I_\lambda = (I + \lambda A)^{-1}$. So the first statement of (A.2) implies $u \in V$. Next noting the inequality (5), we have

$$\begin{aligned} K\|I_\lambda^m u - I_\mu^n u\|^p &\le p(|AI_\lambda^m u| + |AI_\mu^n u|)|I_\lambda^m u - I_\mu^n u|^{p-1} \\ &\le 2pN(u)|I_\lambda^m u - I_\mu^n u|^{p-1} \end{aligned}$$

for $\lambda, \mu > 0$ and positive integers m and n, where $N(u) = \sup_{0<\lambda<1} |\lambda^{-1}(u - I_\lambda u)|$. Hence recalling that V is included continuously in X, letting $\lambda, \mu \downarrow 0$ with $m = [t/\lambda]$ and $n = [s/\mu]$ and using the convergence in X of (3), we can obtain

$$K\|S(t)u - S(s)u\|^p \le 2pN(u)|S(t)u - S(s)u|^{p-1}$$

for $t, s \ge 0$, which implies $S(\cdot)u \in C([0, \infty); V)$ (see [**11**, Lemma 2]). Thus (S.2) holds true for the semigroup. Q.E.D.

EXAMPLES. Let Ω be a bounded domain in R^N with smooth boundary $\partial\Omega$; and let Δ, grad, and $\partial/\partial n$ denote the Laplacian, the gradient, and the outward normal derivative, respectively. A familiar model of the inequality (5) is

$$-\int_\Omega (\Delta u - \Delta v)(u - v)\, dx \ge \int_\Omega |\operatorname{grad}(u - v)|^2\, dx \tag{6}$$

for $u, v \in C^\infty(\Omega)$ such that

$$(\partial u/\partial n - \partial v/\partial n)(u - v) \le 0 \quad \text{on } \partial\Omega.$$

This is a simple consequence of Green's formula.

Let $H^k(\Omega)$ and $H_0^k(\Omega)$ be the usual Sobolev spaces—the real Hilbert spaces with norm $\|\cdot\|_k$ and inner product $(\cdot, \cdot)_k$. Our first example is $-\Delta$ with a nonlinear boundary condition

$$\begin{cases} A_1 u = -\Delta u, \\ D(A_1) = \{u \in H^2(\Omega):\ -\partial u/\partial n \in \beta(u) \text{ a.e. on } \partial\Omega\}; \end{cases}$$

and the second is $-\Delta$ with a nonlinear perturbation

$$\begin{cases} A_2u = -\Delta u + \gamma(u), \\ D(A_2) = \{u \in H^2(\Omega) \cap H_0^1(\Omega)\colon g \in \gamma(u) \text{ a.e. on } \Omega \text{ for some } g \in L^2(\Omega)\}, \end{cases}$$

where β and γ are maximal monotone graphs in $R \times R$ with $0 \in \overline{D(\gamma)}$. As is well known, A_i ($i = 1$ or 2) is maximal monotone in $L^2(\Omega)$, and $\overline{D(A_1)}$ is equal to $L^2(\Omega)$ and $\overline{D(A_2)}$ to the set $\{u \in L^2(\Omega)\colon u \in \overline{D(\gamma)} \text{ a.e. on } \Omega\}$ (see [**2**, pp. 112–116 and **5**, p. 175] for the L^p-theory with $1 \le p \le \infty$). So any $u \in \overline{D(A_i)}$, such that $\|\lambda^{-1}(u - (I+\lambda A_i)^{-1}u)\|_0$ is bounded for $0 < \lambda < 1$, belongs to $D(A_i)$ since $L^2(\Omega)$ is reflexive. In fact, the weak limit w of $\lambda_n^{-1}(u - (I + \lambda_n A_i)^{-1}u)$ as $\lambda_n \downarrow 0$ satisfies

$$(w - A_iv, u - v)_0 \ge 0 \quad \text{for any } v \in D(A_i),$$

which implies $u \in D(A_i)$ by the maximality of A_i (cf. [**7**, pp. 251–252]). On the other hand, the inequality (6) gives

$$(A_iu - A_iv, u - v)_0 \ge \|u - v\|_1^2 - \|u - v\|_0^2 \quad \text{for } u, v \in D(A_i).$$

Thus (A.1) and (A.2) hold true for A_i, or more precisely, for $I + A_i$ with $K = 2$, $p = 2$, and $V = H^1(\Omega)$ ($i = 1$) or $H_0^1(\Omega)$ ($i = 2$). From (S.1) we see, for example, that the semigroup $\{S_i(t)\colon t > 0\}$ generated by A_i satisfies

$$2\int_0^t e^{-2r}\|S_i(r)f - S_i(r)g\|_1^2\,dr \le \|f - g\|_0^2$$

for any $t > 0$ and $f, g \in \overline{D(A_i)}$ ($i = 1$ or 2).

3. Equivalence of conditions. In this section we are concerned with the converse problem. As mentioned in the Introduction, we have to assume that $(X, |\cdot|)$ is a "nice" Banach space; more specifically, X is reflexive and the norm $|\cdot|$ is uniformly Gâteaux differentiable. Generally, the norm $\|\cdot\|$ of a Banach space Y is said to be uniformly Gâteaux differentiable if, for each $y \in U$ where $U = \{x \in Y\colon \|x\| = 1\}$, the limit $\lim_{\lambda\downarrow 0} \lambda^{-1}(\|x + \lambda y\| - \|x\|)$ is approached uniformly as x varies over U. It is known (see [**10**, §1]) that every Banach space with a uniformly convex dual, which includes Hilbert spaces and the L^p-spaces for $1 < p < \infty$, becomes a "nice" Banach space in the above sense.

The following is the main result of this paper.

THEOREM 2. *Assume, in addition, that X is reflexive and the norm $|\cdot|$ is uniformly Gâteaux differentiable and that C is convex. Let $\{S(t)\colon t > 0\}$ be a semigroup of operators mapping C into C with properties* (S.1) *and* (S.2). *Then there exists an accretive subset A of $X \times X$ satisfying* (A.1) *and* (A.2).

PROOF. We shall divide the proof into two steps.

(S.1) $\to$ (A.1). Clearly the condition (S.1) implies that $\{S(t) : t > 0\}$ is a semigroup of contractions from C into C. Therefore, the operator $J_{\lambda,t} = (I - \lambda t^{-1}(I - S(t)))^{-1}$ is well defined for $\lambda > 0$ and $t > 0$, and becomes a contraction from C into C. Baillon [**1**] and Reich [**10**, Theorem 3.1] established the following:

LEMMA 2. *Under the assumptions of Theorem* 2 *except* (S.2), *the following hold:*

(i) *For each $x \in C$ and $\lambda > 0$, the limit $J_\lambda x$ in X of $J_{\lambda,t}x$ does exist as $t \downarrow 0$.*

(ii) $J_\lambda x$ *satisfies, for* $\lambda > 0$ *and* $t > 0$,

$$|t^{-1}(I - S(t))J_\lambda x| \leq |\lambda^{-1}(x - J_\lambda x)|, \tag{7}$$

and converges in X *to* $x \in C$ *as* $\lambda \downarrow 0$.

This lemma implies that the infinitesimal generator A_0:

$$\begin{cases} A_0 u = \lim_{t\downarrow 0} t^{-1}(I - S(t))u, \\ D(A_0) = \{u \in C\colon \lim_{t\downarrow 0} t^{-1}(I - S(t))u \text{ exists}\} \end{cases}$$

is densely defined in C, which was established first by Komura [**6**] in the case of a Hilbert space. Moreover, Baillon [**1**] proved that the closure A in $X \times X$ of the set

$$\bigcup_{\lambda>0} \{[J_\lambda x, \lambda^{-1}(x - J_\lambda x)]\colon x \in C\} \cup A_0$$

becomes an accretive subset of $X \times X$ with the property (A.1), and that the semigroup is represented in terms of A by (3).

(S.1) and (S.2) $\to$ (A.2). Assume that, for $u \in C$, $|\lambda^{-1}(u - (I + \lambda A)^{-1}u)|$ is bounded for $0 < \lambda < 1$. Then, by Lemma 1 with (3), $|t^{-1}(u - S(t)u)|$ is bounded for $0 < t < 1$. Hence, by (S.2), we obtain $u \in V$.

We have now only to show that A defined in the above satisfies the inequality (4). To this end, the following inequality plays an important role:

$$\begin{aligned} &p|x - y|^{p-1}\tau(x - y, t^{-1}(I - S(t))x - t^{-1}(I - S(t))y) \\ &\qquad \geq Kt^{-1} \int_0^t \|S(r)x - S(r)y\|^p \, dr \end{aligned} \tag{8}$$

for $x, y \in C$ and $t > 0$. Indeed, the left-hand side of (8) is not smaller than

$$\begin{aligned} &p|x - y|^{p-1}(t^{-1}|x - y| - t^{-1}|S(t)x - S(t)y|) \\ &\qquad \geq t^{-1}|x - y|^p - t^{-1}|S(t)x - S(t)y|^p \end{aligned}$$

by Young's inequality. Thus (1) implies (8).

The following is our key lemma.

LEMMA 3. *Under the assumption of Lemma* 2,

$$t^{-1} \int_0^t \|S(r)J_{\lambda,t}x - S(r)J_\lambda x\|^p \, dr \to 0 \qquad \text{as } t \downarrow 0$$

for each $x \in C$ *and* $\lambda > 0$.

PROOF. Replacing x and y in (8) by $J_{\lambda,t}x$ and $J_\lambda x$, we have that

$$\begin{aligned} Kt^{-1} &\int_0^t \|S(r)J_{\lambda,t}x - S(r)J_\lambda x\|^p \, dr \\ &\leq p|J_{\lambda,t}x - J_\lambda x|^{p-1}(|t^{-1}(I - S(t))J_{\lambda,t}x| + |t^{-1}(I - S(t))J_\lambda x|) \\ &\leq p|J_{\lambda,t}x - J_\lambda x|^{p-1}(|\lambda^{-1}(x - J_{\lambda,t}x)| + |\lambda^{-1}(x - J_\lambda x)|), \end{aligned}$$

where we have used (7). Letting $t \downarrow 0$ and using Lemma 2, (i), we obtain the lemma. Q.E.D.

We are now in a position to show (4) for our A. In view of the definition of A, we find that it suffices to verify (5) in the following three cases.

First, let u and v be in $D(A_0)$. Then, by (S.2), $S(\cdot)u$ and $S(\cdot)v$ belong to $C([0,\infty);V)$. Therefore, letting $t \downarrow 0$ in the inequality (8) with x and y replaced by u and v, and noting the upper semicontinuity of $\tau(\cdot,\cdot)$, we get

$$p|u-v|^{p-1}\tau(u-v,\ A_0u-A_0v) \geq K\|u-v\|^p.$$

Next, replace x and y in (8) by $u \in D(A_0)$ and $J_{\lambda,t}x$. Then we see that, for $x \in C$ and $\lambda > 0$,

$$\begin{aligned}
p|u-J_{\lambda,t}x|^{p-1}&\tau(u-J_{\lambda,t}x, t^{-1}(I-S(t))u-\lambda^{-1}(x-J_{\lambda,t}x)) \\
&\geq Kt^{-1}\int_0^t \|S(r)u-S(r)J_{\lambda,t}x\|^p\,dr \\
&\geq K\left\{\left(t^{-1}\int_0^t \|S(r)u-S(r)J_\lambda x\|^p\,dr\right)^{1/p}\right. \\
&\qquad \left.-\left(t^{-1}\int_0^t \|S(r)J_\lambda x-S(r)J_{\lambda,t}x\|^p\,dr\right)^{1/p}\right\}^p
\end{aligned}$$

by Minkowski's inequality. Here we note that, by (S.2) and (7), $J_\lambda x$ belongs to V and $S(\cdot)J_\lambda x$ to $C([0,\infty);V)$. Taking the limit as $t\downarrow 0$ and using Lemma 3, we obtain

$$p|u-J_\lambda x|^{p-1}\tau(u-J_\lambda x, A_0u-\lambda^{-1}(x-J_\lambda x)) \geq K\|u-J_\lambda x\|^p.$$

Finally using (8) for $J_{\lambda,t}x$ and $J_{\mu,t}y$ instead of x and y, we have for $x, y \in C$ and $\lambda, \mu > 0$

$$\begin{aligned}
p|J_{\lambda,t}x-J_{\mu,t}y|^{p-1}&\tau(J_{\lambda,t}x-J_{\mu,t}y, \lambda^{-1}(x-J_{\lambda,t}x)-\mu^{-1}(y-J_{\mu,t}y)) \\
\geq K&\left\{\left(t^{-1}\int_0^t \|S(r)J_\lambda x-S(r)J_\mu y\|^p\,dr\right)^{1/p}\right. \\
&-\left(t^{-1}\int_0^t \|S(r)J_\lambda x-S(r)J_{\lambda,t}x\|^p\,dr\right)^{1/p} \\
&\left.-\left(t^{-1}\int_0^t \|S(r)J_\mu y-S(r)J_{\mu,t}y\|^p\,dr\right)^{1/p}\right\}^p.
\end{aligned}$$

Going to the limit as $t\downarrow 0$, we conclude

$$p|J_\lambda x-J_\mu y|^{p-1}\tau(J_\lambda x-J_\mu y, \lambda^{-1}(x-J_\lambda x)-\mu^{-1}(y-J_\mu y)) \geq K\|J_\lambda x-J_\mu y\|^p.$$

Thus our operator A satisfies the desired inequality (4) or (5), and hence the conditions (A.2). Q.E.D.

We shall finish our discussion with the linear case. In this case the "nice" property of $(X, |\cdot|)$ may be dropped and the situation becomes simpler. The results of Theorems 1 and 2 can be formulated as follows.

COROLLARY. *Let A be a linear m-accretive operator such that $D(A)$ is included in V and dense in X. Let $\{S(t)\colon t>0\}$ be the strongly continuous semigroup in X generated by A. Then*

$$|u|^p + K\lambda\|u\|^p \leq |u+\lambda Au|^p \tag{9}$$

for $\lambda > 0$ *and* $u \in D(A)$, *if and only if*

$$S(\cdot)u \in C([0,\infty);V) \quad \textit{for } u \in D(A);$$
$$S(\cdot)x \in L^p(0,t;V) \quad \textit{for } x \in X \textit{ and } t > 0$$

with $|S(t)x|^p + K\int_0^t \|S(r)x\|^p\, dr \leq |x|^p$.

REMARK. In this linear case, let A replace $bI+A$ for a real number b if necessary, and $D(A^\alpha)$ denote the domain of fractional power A^α, $0 < \alpha < 1$, with the graph norm. Recall here that A^α is defined by $A^\alpha = (A^{-\alpha})^{-1}$ with

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda I + A)^{-1}\, d\lambda.$$

Then (9) with $V = D(A^\alpha)$ implies by the corollary that, for each $x \in X$, $S(t)x$ belongs to $D(A^\alpha)$ for a.a. $t > 0$ and hence to $D(A)$ for a.a. $t > 0$. Thus for each $x \in X$ and positive integer n, $S(t)x$ belongs to $D(A^n)$ for a.a. $t > 0$. Note that our method does not require the use of complex numbers differently from Pazy's approach [**8**] to semigroups of C^∞.

EXAMPLE. Let us recall the operator A_2 given in §2 and consider, in particular, the natural realization in $L^2(\Omega)$ of $-\Delta$: $A_3u = -\Delta u$ with $D(A_3) = H^2(\Omega)\cap H_0^1(\Omega)$. Then the equality $(A_3u, u)_0 = \|u\|_1^2 - \|u\|_0^2$ implies the coincidence $D((I + A_3)^{1/2}) = H_0^1(\Omega)$ and the estimate $(1 - 2\lambda)\|u\|_0^2 + 2\lambda\|(I + A_3)^{1/2}u\|_0^2 \leq \|u + \lambda A_3u\|_0^2$ for $0 < \lambda < 1/2$ and $u \in D(A_3)$. We can thus deduce without the use of complex numbers that the semigroup $\{S_3(t) \colon t > 0\}$ generated by A_3 is very smooth in the above sense.

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