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MARTIN BOUNDARIES OF DENJOY DOMAINS

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ABSTRACT. Let E ($\subset \hat{\mathbf{C}}$) be a compact set in the real axis. It is shown that there exists an E with zero linear measure such that Martin compactification of the domain $\hat{\mathbf{C}} - E$ is not homeomorphic to $\hat{\mathbf{C}}$. Moreover, it is shown that if for some $\lambda > \frac{1}{2}$

$$\frac{|E^c \cap [-t,t]|}{t} = O\left(\left(\frac{1}{\log t^{-1}}\right)^{\lambda}\right) \qquad (t \to 0),$$

the set of minimal Martin boundary points of $\hat{\mathbf{C}} - E$ 'over 0' consists of two points. This assertion is not valid for $\lambda = \frac{1}{2}$.

Consider a domain D in $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ such that $\hat{\mathbf{C}} - D \subset \hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. Such a domain D is said to be a *Denjoy domain* (cf. Garnett and Jones [**4**]). For $p \in E = \partial D$, let $\mathcal{P}_p = \mathcal{P}_p(D)$ be the class of positive harmonic functions on D which are bounded except for any neighborhood of p, and have vanishing boundary values at every regular point of E except p. Denote by $\dim \mathcal{P}_p$ the cardinal number of the set of minimal functions h in $\mathcal{P}_p$ satisfying the normalized condition $h(a) = 1$, $a \in D$. In terms of Martin compactification, $\dim \mathcal{P}_p$ means the cardinal number of the set of minimal boundary points 'over p'. It is easily seen that $\dim \mathcal{P}_p \geq 1$ for every $p \in E$ (cf. e.g. Benedicks [**2**]). Ancona [**1**] and Benedicks [**2**], independently, showed that $\dim \mathcal{P}_p \leq 2$ for every $p \in E$. Also, Maitani showed that if $\dim \mathcal{P}_p = 1$ for every $p \in E$ (in this case, the Martin compactification D^* of D is homeomorphic to $\overline{D}$, the closure of D in $\hat{\mathbf{C}}$) the linear measure $|E|$ of E is zero.

In this paper, applying the Benedicks criterion in [**2**], we shall show that the converse of the above Maitani result is invalid (§1), and study the cardinal number $\dim \mathcal{P}_p$ for p which is a point of density of E (§2).

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1. We start by restating Maitani's result in the introduction.

THEOREM 1. *Let D be a Denjoy domain. Suppose that* $\dim \mathcal{P}_p(D) = 1$ *for every $p \in E = \partial D$. Then the linear measure $|E|$ of E is zero.*

Since the above result was unpublished, we give the proof for the sake of completeness.

If D is of null boundary, the capacity of E is zero, and hence $|E| = 0$. Thus we may assume that there exists the Green's function g on D. Choose a point

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$a \in D \cap \mathbf{R}$. By the assumption, the limit

$$\lim_{\varsigma \to p} \frac{g(z,\varsigma)}{g(a,\varsigma)} \quad (= k_p(z))$$

exists for every $p \in E$. From the fact that $g(a,\bar{\varsigma}) = g(a,\varsigma)$, it follows that

$$k_p(\bar{z}) = \lim_{\varsigma \to p} \frac{g(\bar{z},\varsigma)}{g(a,\varsigma)} = \lim_{\bar{\varsigma} \to p} \frac{g(z,\bar{\varsigma})}{g(a,\bar{\varsigma})} = k_p(z).$$

Hence, by general Martin theory (here, we identify E to the Martin boundary of D), it is seen that

$$h(\bar{z}) = \int_E k_p(\bar{z})\, d\mu_h(p) = \int_E k_p(z)\, d\mu_h(p) = h(z)$$

for every positive harmonic function h on D, where μ_h is a positive measure on E. This implies that $u(\bar{z}) = u(z)$ for every bounded harmonic function u on D, and hence it is easily verified that $|E| = 0$. □

Here we recall Benedicks' result in [**2**]. Let E be a compact subset of $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ containing ∞. Denote by $Q(t,r)$, $t \in \mathbf{R}$, the square $\{\xi + i\eta; |\xi - t| < r/2, |\eta| < r/2\}$. For an arbitrarily fixed α in the interval $(0,1)$ and every x in $\mathbf{R}$, let $\beta_x(\cdot) = \beta_x(\cdot; E, \alpha)$ be the solution of Dirichlet problem on $Q(x, \alpha|x|) - E$ for boundary values $\beta_x = 1$ on $\partial Q(x,\alpha|x|)$ and $\beta_x = 0$ on $E \cap Q(x,\alpha|x|)$. Then, Benedicks showed the following:

BENEDICKS' CRITERION I. *For every α with $0 < \alpha < 1$,*

$$\dim \mathcal{P}_\infty(\hat{\mathbf{C}} - E) = 1 \quad \textit{if and only if} \quad \int_{|x|\geq 1} \frac{\beta_x(x)}{|x|}\, dx = \infty,$$

$$\dim \mathcal{P}_\infty(\widehat{\mathbf{C}} - E) = 2 \quad \textit{if and only if} \quad \int_{|x|\geq 1} \frac{\beta_x(x)}{|x|}\, dx < \infty.$$

We are in the stage to prove the following

THEOREM 2. *Let E_0 be a closed set in the interval $[-\frac{1}{2}, \frac{1}{2}]$ of positive capacity. Set*

$$E_n = E_0 + n = \{x + n; x \in E_0\} \qquad (n \in \mathbf{Z})$$

and

$$E = \left(\bigcup_{n=-\infty}^{\infty} E_n \right) \cup \{\infty\}.$$

Then, for the Denjoy domain $D = \widehat{\mathbf{C}} - E$, $\dim \mathcal{P}_\infty(D) = 2$.

Immediately, Theorem 2 shows that the converse of Theorem 1 is invalid.

LEMMA 3. *Let E be as in Theorem 2 and h_n $(n \in \mathbf{N})$ be the solution of Dirichlet problem on $Q(0,3^n) - E$ for boundary values $h_n = 1$ on $\partial Q(0,3^n)$ and $h_n = 0$ on $E \cap Q(0,3^n)$. Then, there exists a constant c in $(0,1)$ such that*

$$h_n(x) \leq c^n \qquad \left(x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right), \tag{1}$$

where c is independent of n and x.

PROOF. Set $c_1 = \sup_{x \in [-1/2, 1/2]} h_1(x)$. Then $0 < c_1 < 1$ and

$$h_n(x) \leq c_1 \qquad (x \in [-3^{n-1}/2, 3^{n-1}/2]). \tag{2}$$

Let h be the harmonic function on $Q(0,3) - [-\frac{1}{2}, \frac{1}{2}]$ with boundary values $h = 1$ on $\partial Q(0,3)$ and $h = c_1$ on $[-\frac{1}{2}, \frac{1}{2}]$. Observe that

$$h_n(z) \le h(z/3^{n-1}) \qquad (z \in Q(0,3^n)). \tag{3}$$

Set $c = \sup_{z \in Q(0,1)} h(z)$. Obviously $c_1 < c < 1$. By (3), we see that $h_n(z) \le c$ on $Q(0,3^{n-1})$, and hence $h_n(z) \le c h_{n-1}(z)$ on $Q(0,3^{n-1})$. Therefore, (2) implies that

$$h_n(x) \le c c_1 \qquad (x \in [-3^{n-2}/2, 3^{n-2}/2]).$$

From this, it follows that $h_n(z) \le c h(z/3^{n-2})$ on $Q(0,3^{n-1})$, and hence, by the definition of c,

$$h_n(z) \le c^2 \qquad (z \in Q(0,3^{n-2})).$$

Repeating this argument, we conclude that

$$h_n(z) \le c^n \qquad (z \in Q(0,1)). \quad \square$$

PROOF OF THEOREM 2. Let $\alpha = \frac{2}{3}$. For every $x \in \mathbf{R}$ with $3^{n+1} \le |x| \le 3^{n+2}$, choose $m \in \mathbf{Z}$ such that $|x - m| \le \frac{1}{2}$. Since $Q(0,3^n) + m = Q(m,3^n) \subset Q(x, \alpha|x|)$,

$$\beta_x(z) \le h_n(z - m) \qquad (z \in Q(m,3^n)).$$

With (1), this implies that

$$\beta_x(x) \le h_n(x - m) \le c^n \qquad (x \in \mathbf{R},\ 3^{n+1} \le |x| \le 3^{n+2}).$$

Therefore,

$$\begin{aligned}\int_{|x|\ge 3} \frac{\beta_x(x)}{|x|}\,dx &= \sum_{n=0}^{\infty} \int_{3^{n+1}\le|x|\le 3^{n+2}} \frac{\beta_x(x)}{|x|}\,dx \\ &\le 2\sum_{n=0}^{\infty} \frac{c^n}{3^{n+1}} 3^{n+2} < \infty.\end{aligned}$$

By Benedicks' criterion I, this completes the proof. $\square$

REMARK. We take the Cantor ternary set for E_0 in Theorem 2. Then, for the resulting Denjoy domain D, a boundary Harnack inequality yields that $\dim \mathcal{P}_p(D) = 1$ for every $p \in E - \{\infty\}$, however $\dim \mathcal{P}_\infty(D) = 2$ by Theorem 2.

2. Throughout this section, suppose that E is a compact set in $\mathbf{R}$ such that 0 is a point of density of E. We shall study the cardinal number $\dim \mathcal{P}_0(D)$, where $D = \hat{\mathbf{C}} - E$.

Denote by $B(t,r)$ the disk $\{z; |z - t| < r\}$. For an arbitrarily fixed $\delta \in (0, \frac{1}{3})$ and every $x \in \mathbf{R} - \{0\}$, let $\gamma_x(\cdot) = \gamma_x(\cdot; E, \delta)$ be the solution of Dirichlet problem on $B(x, \delta|x|) - E$ for boundary values $\gamma_x = 1$ on $\partial B(x, \delta|x|)$ and $\gamma_x = 0$ on $E \cap B(x, \delta|x|)$. Choose α_1 and α_2 in $(0,1)$ such that $0 < 2\alpha_1 < \delta < \alpha_2/(2 + \alpha_2)$. Consider the mapping $\phi(z) = 1/z$. Set $\omega_x^i = \beta_{1/x}(\cdot; \phi(E), \alpha_i)$, $i = 1, 2$. Note that

$$Q(1/x, \alpha_1/|x|) \subset \phi(B(x, \delta|x|)) \subset Q(1/x, \alpha_2/|x|).$$

Therefore, $\omega_x^2(1/x) \le \gamma_x(x) \le \omega_x^1(1/x)$, and hence, by Benedicks' criterion I, we obtain the following.

BENEDICKS' CRITERION II. *For every* $\delta \in (0, \frac{1}{3})$,

$$\dim \mathcal{P}_0(D) = 1 \text{ if and only if } \int_{|x|\le 1} \frac{\gamma_x(x)}{|x|}\,dx = \infty,$$

$$\dim \mathcal{P}_0(D) = 2 \text{ if and only if } \int_{|x|\le 1} \frac{\gamma_x(x)}{|x|}\,dx < \infty.$$

We shall prove the following

THEOREM 4. *Suppose that there exists* $\lambda > \frac{1}{2}$ *such that*

$$\frac{|E^c \cap I_t|}{t} = O\left(\left(\frac{1}{\log t^{-1}}\right)^{\lambda}\right),$$

where $I_t = [-t, t]$. *Then* $\dim \mathcal{P}_0(D) = 2$.

Let B be the unit disk $B(0,1)$. For $r \in (0,1]$, denote by Φ_r the class of harmonic functions u such that (i) there exists a closed set E_u in $[-1,1]$ with $|E_u| \ge 2(1-r)$, and (ii) u is the solution of Dirichlet problem on $B - E_u$ for boundary values $u = 1$ on ∂B and $u = 0$ on E_u. For the proof of Theorem 4, we need following lemmas.

LEMMA 5. *Suppose that* $u \in \Phi_r$. *Then* $u(ti)$ *is an increasing function of* $t \in (0,1)$.

PROOF. We may assume that E_u consists of finite number of closed intervals. Let g_B be the Green's function on B and set $\hat{g}(z,ti) = g_B(z,ti) + g_B(z,-ti)$, $0 < t < 1$. Applying Green's formula to $1-u$ and $\hat{g}(z,ti)$ on $B^+ = B \cap \{\operatorname{Im} z > 0\}$, we see that

$$1 - u(ti) = \frac{1}{2\pi}\int_{-1}^{1} \hat{g}(x,ti)\left(\frac{\partial u}{\partial y}\right)_{y=0} dx, \tag{4}$$

where $z = x + iy$. Observe that $\hat{g}(x,ti)$, $-1 < x < 1$, are decreasing functions of $t \in (0,1)$ and $(\partial u/\partial y)_{y=0} \ge 0$. Hence, by (4), $u(ti)$ is increasing on $(0,1)$. □

LEMMA 6. *For each* $\rho \in [0,1)$, *there exists a constant* C_ρ *depending only on* ρ *such that*

$$u(0) \le C_\rho r^\rho, \qquad 0 < r \le 1, \tag{5}$$

where u *is any function in* Φ_r *such that* 0 *is a regular boundary point or an interior point of* $B - E_u$.

PROOF. Let g^+ be the Green's function on $B^+ = B \cap \{\operatorname{Im} z > 0\}$. Since $g^+(z,w) = g_B(z,w) - g_B(z,\bar{w})$, $z = x + iy$,

$$\left(\frac{\partial g^+(z,ti)}{\partial y}\right)_{y=0} \le \frac{2t}{x^2+t^2}, \qquad 0 < t < 1. \tag{6}$$

Let U be the harmonic function on B^+ with boundary values $U = 1$ on $\partial B^+ \cap \{|z| = 1\}$ and $U = 0$ on $\partial B^+ \cap \mathbf{R}$. Observe that

$$U(ti) = (4/\pi)\tan^{-1} t, \qquad 0 < t < 1. \tag{7}$$

In order to show (5), it is sufficient to show that, for each $n \in \{0\} \cup \mathbf{N}$, there exists a constant C_n depending only on n such that

$$u(0) \le C_n r^{1-2^{-n}}. \tag{8}$$

When $n = 0$, (8) is trivial. Suppose that (8) is valid for some n. For each $u \in \Phi_r$ and each $x \in [-\frac{1}{2}, \frac{1}{2}]$, there exists a $u_x \in \Phi_{r_1}$, $r_1 = \min(1, 2r)$, such that $u(x + z/2) \le u_x(z)$, $z \in B$. Therefore,

$$u(x) \le u_x(0) \le C_n r_1^{1-2^{-n}} = C_n r^{1-2^{-n}} \tag{9}$$

for almost all $x \in [-\frac{1}{2}, \frac{1}{2}]$. Then, (6), (7), (9), and Green's formula yield that, for $t \in (0, 1)$,

$$\begin{aligned}
u(ti) &= U(ti) + (u - U)(ti) \\
&= U(ti) + \frac{1}{2\pi} \int_{-1}^{1} u(x) \left(\frac{\partial g^+(z, ti)}{tz\partial y} \right)_{y=0} dx \\
&\le Ct + \frac{1}{\pi} \left(\int_{-1/2}^{1/2} \frac{tu(x)}{x^2 + t^2}\, dx + \int_{[-1,1]-[-1/2,1/2]} \frac{tu(x)}{x^2 + t^2}\, dx \right) \\
&\le Ct + \frac{1}{\pi} \left(\frac{C_n r^{1-2^{-n}}}{t} \int_{[-1/2,1/2]\cap E_u^c} dx + 4 \int_{E_u^c} dx \right) \\
&\le Ct + \frac{1}{\pi} \left(\frac{C_n r^{1-2^{-n}}}{t} 2r + 8r \right).
\end{aligned}$$

Therefore, putting $t = r^{1-2^{-n-1}}$ and, by Lemma 5, we have

$$u(0) \le u(r^{1-2^{-n-1}} i) \le C_{n+1} r^{1-2^{-n-1}}$$

if 0 is a regular boundary point or an interior point of $B - E_u$. □

PROOF OF THEOREM 4. There exists a constant C independent of t such that

$$|E^c \cap I_t| \le Ct/(\log t^{-1})^{\lambda}, \qquad 0 < t \le 1. \tag{10}$$

Then, for each x in $(0, \frac{1}{2}]$,

$$\frac{1}{\delta x} |E^c \cap [x - \delta x, x + \delta x]| \le \frac{1}{\delta x} \frac{C(x + \delta x)}{(\log(x + \delta x)^{-1})^{\lambda}} \le \frac{C}{(\log x^{-1})^{\lambda}}. \tag{11}$$

Put $r = r_x = C/(2(\log(1/x))^{\lambda})$ and choose $t_0 \in (0, \frac{1}{2}]$ such that $r_x \le 1$ for each $x \in (0, t_0]$. Consider the function $u(z) = \gamma_x(x + \delta x z)$ on B. From (11), it follows that $u \in \Phi_{r_x}$ for each $x \in (0, t_0]$. Hence, by Lemma 6, we can choose a $\rho \in (0, 1)$ such that $\lambda\rho > \frac{1}{2}$ and

$$\gamma_x(x) = u(0) \le C_\rho r_x^{\rho} = C/(\log x^{-1})^{\lambda\rho} \tag{12}$$

for almost all x in $(0, t_0]$, where C is independent of x. Note that $\sigma = \lambda\rho + \lambda > 1$. Setting $X(x) = \int_{E^c \cap [0,x]} dt$, by (10),

$$X(x) \le Cx/(\log x^{-1})^{\lambda}. \tag{13}$$

Write $t_1 = \min(t_0, 1/e^{\lambda\rho})$. Since $\gamma_x(x) = 0$ almost everywhere on E, using (12) and (13), we see that

$$\begin{aligned}\int_0^{t_1} \frac{\gamma_x(x)}{x}\,dx &\le \int_0^{t_1} \frac{C}{x(\log x^{-1})^{\lambda\rho}}\,dX \\ &= \frac{CX(t_1)}{t_1(\log t_1^{-1})^{\lambda\rho}} - \int_0^{t_1} CX(x)\,d\left(\frac{1}{x(\log x^{-1})^{\lambda\rho}}\right) \\ &\le C + C\int_0^{t_1} \frac{x}{(\log x^{-1})^{\lambda}}\frac{1}{x^2(\log x^{-1})^{\lambda\rho}}\left(1 - \frac{\lambda\rho}{\log x^{-1}}\right)dx \\ &\le C + C\int_0^{t_1} \frac{dx}{x(\log x^{-1})^{\sigma}} < \infty.\end{aligned}$$

Similar argument yields that $\int_{[-1,0)}(\gamma_x(x)/|x|)\,dx < \infty$, and therefore the criterion II completes the proof. □

3. In this section, we shall give an example of a compact subset E of $\mathbf{R}$ such that

$$|E^c \cap I_t|/t = O\left((\log t^{-1})^{-1/2}\right)$$

and $\dim \mathcal{P}_0(\hat{\mathbf{C}} - E) = 1$, which shows that the condition $\lambda > \frac{1}{2}$ in Theorem 4 is, in a sense, best possible.

LEMMA 7. *For r in $(0,1)$, let u_r be the harmonic function on $B - ([-1,-r] \cup [r,1])$ with boundary values $u_r = 1$ on ∂B and $u_r = 0$ on $[-1,-r] \cup [r,1]$. Then there exists a constant C independent of r such that $u_r(0) \ge Cr$.*

PROOF. Consider two functions $f(z)$ and $g(w)$ such that

$$f(z) = \frac{\sqrt{(z+r)/(1+rz)} - \sqrt{r}}{1 - \sqrt{r(z+r)/(1+rz)}} \cdot \frac{1 + \sqrt{r(z+r)/(1+rz)}}{\sqrt{(z+r)/(1+rz)} + \sqrt{r}}$$

and

$$g(w) = \frac{\sqrt{(c-w)/(1-cw)} - \sqrt{c}}{1 - \sqrt{c(c-w)/(1-cw)}} \cdot \frac{1 + \sqrt{c(c-w)/(1-cw)}}{\sqrt{(c-w)/(1-cw)} + \sqrt{c}},$$

where $c = f(r)$. Then, it is not difficult to see that $g(f(z))$ is the conformal mapping of $B - ([-1,-r] \cup [r,1])$ to B. Computing the arc length of $g(f(\partial B))$, we have

$$u_r(0) = \frac{2}{\pi}\sin^{-1}\frac{6c - c^2 - 1}{(1+c)^2} \ge Cr. \quad \square$$

Let $E = [-1,1] - \bigcup_{n=2}^{\infty} J_n$, where J_n is the open interval $((1 - 1/\sqrt{n})e^{-n}, (1 + 1/\sqrt{n})e^{-n})$, $n = 2, 3, \ldots$. Then, for each $x \in J_n' = ((1 - 1/2\sqrt{n})e^{-n}, (1 + 1/2\sqrt{n})e^{-n})$, there exists a constant C independent of n such that

$$(1/\delta x)\,\mathrm{dist}(x, E) \ge C/\sqrt{n}.$$

By the definition of u_r in Lemma 7, this means that $\gamma_x(x + \delta xz) \ge u_{r_n}(z)$, $z \in B$, where $r_n = C/\sqrt{n}$. Hence, from Lemma 7, it follows that

$$\gamma_x(x) \ge u_{r_n}(0) \ge C/\sqrt{n}, \qquad x \in J_n'.$$

Therefore,

$$\int_0^1 \frac{\gamma_x(x)}{x}\,dx \geq \sum_{n=2}^{\infty} \int_{J'_n} \frac{\gamma_x(x)}{x}\,dx$$
$$\geq \sum_{n=2}^{\infty} \frac{Ce^{n+1}}{\sqrt{n}} \frac{e^{-n}}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{C}{n} = \infty.$$

Thus the criterion II shows that $\dim \mathcal{P}_0(\hat{\mathbf{C}} - E) = 1$. On the other hand, it is easily seen that $|E^c \cap I_t|/t = O((\log t^{-1})^{-1/2})$.

REMARK. It seems that the function u_r in Lemma 7 satisfies that $u_r(0) = \sup_{u \in \Phi_r} u(0)$, although we have not succeeded in the proof.

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