TOTAL PARACOMPACTNESS AND BANACH SPACES
FRANCISCO GALLEGO LUPIÁNEZ

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ABSTRACT. In this paper, we study some problems related to the Corson theorem. In particular we prove that $c_0$ does not fulfil such a theorem; hence this theorem is not valid for all infinite-dimensional Banach spaces. We give also generalizations of Corson’s theorem for some infinite-dimensional normed spaces.

A topological space is said to be totally paracompact [5] if every open basis contains a locally finite covering. It is known [4] that “every totally paracompact complete metric space is $C$-scattered” and “every $\sigma$-locally compact paracompact space is totally paracompact”. Then, every Banach space is totally paracompact if and only if it is finite dimensional. Thus, every infinite-dimensional Banach space necessarily has an open basis which contains no locally finite covering (i.e. a coarse open basis [3]). We raise the question of how to give an intrinsic description of those infinite-dimensional Banach spaces such that the open basis formed by all open balls is coarse. Corson’s theorem is related with this question:

THE CORSON THEOREM [2]. For any covering $\mathcal{U}$ of a reflexive, infinite-dimensional Banach space $B$, where $\mathcal{U}$ is formed by bounded, convex sets, there is a point $x$ in $B$ such that each neighborhood of $x$ meets infinitely many elements of $\mathcal{U}$; i.e., $\mathcal{U}$ is not locally finite.

In this paper, we study some problems related to the Corson theorem. In particular, we shall prove that $c_0$ does not fulfil such a theorem, hence this theorem is not valid for all infinite-dimensional Banach spaces. We shall give a generalization of the Corson theorem for some infinite-dimensional normed spaces.

THEOREM 1. For every $r > 0$, there exists an open covering of $c_0$, which is locally finite and is formed by balls of radius $r$.

PROOF. Let $\mathcal{U} = \{B_1(0)\} \cup \mathcal{U}_+ \cup \mathcal{U}_-$, where

$\mathcal{U}_+ = \{B_1(x_1, \ldots, x_{i-1}, n + \frac{1}{2}, 0, \ldots) | i \in \mathbb{N}, n \in \mathbb{N}, x_k \in \{0\} \cup \{p + \frac{1}{2}, -q - \frac{1}{2} | p, q \in \mathbb{N}\}, k \in \{1, \ldots, i - 1\}\}$,

$\mathcal{U}_- = \{B_1(y_1, \ldots, y_{j-1}, -m - \frac{1}{2}, 0, \ldots) | j \in \mathbb{N}, m \in \mathbb{N}, y_i \in \{0\} \cup \{p + \frac{1}{2}, -q - \frac{1}{2} | p, q \in \mathbb{N}\}, i \in \{1, \ldots, j - 1\}\}$.
We shall prove that

(1) \( \mathcal{U} \) covers \( c_0 \).

(2) \( \mathcal{U} \) is locally finite in \( c_0 \).

(1) For each \((x_n)_{n \in \mathbb{N}} \in c_0\) there is the first natural number \( n_0 \) such that \(|x_n| < 1\) for every \( n \geq n_0 \).

If \( n_0 = 1 \), we have \(|x_n| < 1\) for every \( n \geq 1 \). Then \( \|(x_n)_{n \in \mathbb{N}}\|_\infty < 1 \) and \((x_n)_{n \in \mathbb{N}} \in B_1(0) \in \mathcal{U} \).

If \( n_0 > 1 \), we have that \(|x_{n_0-1}| \geq 1 \) and

- there is \( m = E[x_{n_0-1}] \in \mathbb{N} \) such that \(|m + \frac{1}{2} - x_{n_0-1}| \leq \frac{1}{2} < 1\) for \( x_{n_0-1} \geq 1 \);
- there is \( m = E[-x_{n_0-1}] \in \mathbb{N} \) such that \(|-m - \frac{1}{2} - x_{n_0-1}| \leq \frac{1}{2} < 1\) for \( x_{n_0-1} \leq -1 \).

(We write \( E[x] \) for the “integral part of \( x \),” the largest integer which does not exceed \( x \)).

Let \((y_n)_{n \in \mathbb{N}} \in c_0 \) be such that \( y_n = 0 \) for every \( n \geq n_0 \),

\[
y_{n_0-1} = \begin{cases} 
  m + \frac{1}{2} & \text{for } x_{n_0-1} \geq 1, \\
  -m - \frac{1}{2} & \text{for } x_{n_0-1} \leq -1,
\end{cases}
\]

for each \( n < n_0 - 1 \), and

\[
y_n = \begin{cases} 
  0 & \text{for } |x_n| < 1, \\
  E[x_n] + \frac{1}{2} & \text{for } x_n \geq 1, \\
  -E[-x_n] - \frac{1}{2} & \text{for } x_n \leq -1.
\end{cases}
\]

Then \( \|(x_n)_{n \in \mathbb{N}} - (y_n)_{n \in \mathbb{N}}\|_\infty < 1 \) and \((x_n)_{n \in \mathbb{N}} \in B_1((y_n)_{n \in \mathbb{N}}) \in \mathcal{U}_+ \cup \mathcal{U}_- \subset \mathcal{U} \).

(2) We shall prove that \( \mathcal{U}_+ \) is locally finite in \( c_0 \) (for \( \mathcal{U}_- \) it is analogous) and clearly we shall have that \( \mathcal{U} \) is locally finite in \( c_0 \).

If \( \mathcal{U}_+ \) is not locally finite in \( c_0 \), there exists \((t_k)_{k \in \mathbb{N}} \in c_0 \) such that for each \( \delta > 0 \) there is an infinite family \( \mathcal{F}_\delta \), which is contained in \( \mathcal{U}_+ \) such that \( B_\delta((t_k)_{k \in \mathbb{N}}) \cap B \neq \emptyset \) for every \( B \in \mathcal{F}_\delta \); then the set

\[
C_\delta = \{ i \in \mathbb{N} \mid \text{there is } m \in \mathbb{N}, \text{there are } x_k \in \{0\} \cup \{p + \frac{1}{2}, -q - \frac{1}{2} | p, q \in \mathbb{N}\} \\
\quad \text{for each } k \in \{1, \ldots, i - 1\} \\
\quad \text{such that } B_1(x_1, \ldots, x_{i-1}, m + \frac{1}{2}, 0, \ldots) \in \mathcal{F}_\delta \}
\]

is nonvoid. Let \( 0 < \delta < \frac{1}{2} \) in the following argument. \( C_\delta \) can be infinite or finite.

If \( C_\delta \) is infinite, then \( C_\delta \supset \{i_n | n \in \mathbb{N}\} \) such that \( i_n < i_{n+1} \) for every \( n \in \mathbb{N} \). Then there exist integers such that

\[
\{B_1(x_1^n, \ldots, x_{i_n-1}^n, m + \frac{1}{2}, 0, \ldots)| n \in \mathbb{N}\} \in \mathcal{F}_\delta
\]

and for every \( n \in \mathbb{N} \), there exists \((z_k^n)_{k \in \mathbb{N}} \in c_0 \) such that

\[
(z_k^n)_{k \in \mathbb{N}} \in B_\delta((t_k)_{k \in \mathbb{N}}) \cap B_1(x_1^n, \ldots, x_{i_n-1}^n, m + \frac{1}{2}, 0, \ldots).
\]

Thus, from \(|z_k^n - m_n - \frac{1}{2}| < 1\) and \(|z_k^n - t_{i_n}| < \delta\) for every \( n \in \mathbb{N} \) it follows that \( m_n - \frac{1}{2} < t_{i_n} + \delta \) for every \( n \in \mathbb{N} \). Since \( m_n \in \mathbb{N} \) for every \( n \in \mathbb{N} \), we have that \( t_{i_n} > \frac{1}{2} - \delta \) for every \( n \in \mathbb{N} \). Then \((t_k)_{k \in \mathbb{N}} \) is not an element of \( c_0 \) and this is a contradiction.

If \( C_\delta \) is finite, let an \( i_\delta \in C_\delta \). Then

\[
B_\delta((t_k)_{k \in \mathbb{N}}) \cap B_1(x_1, \ldots, x_{i_\delta-1}, m + \frac{1}{2}, 0, \ldots) \neq \emptyset
\]

for infinites \((x_1, \ldots, x_{i_\delta-1}, m + \frac{1}{2})\), where \( x_k \in \{0\} \cup \{p + \frac{1}{2}, -q - \frac{1}{2} | p, q \in \mathbb{N}\} \) for each \( k \in \{1, \ldots, i_\delta - 1\} \) and \( m \in \mathbb{N} \). Then, there exists an infinite family \( \mathcal{G}_\delta \subset \mathcal{F}_\delta \).
formed by balls $B_1(x^n_1, \ldots, x^n_{i_\delta-1}, m_n + \frac{1}{2}, 0, \ldots)$ which, we suppose, are pairwise different.

Now, for $i_\delta$ let $U(2, i_\delta - 1)$ be the set of all $(i_\delta - 1)$-samples with repetitions of the elements 0 and 1.

For each $s \in U(2, i_\delta - 1)$, let $c_s = \{k \in \{1, \ldots, i_\delta - 1\} | s(k) \neq 0\}$.

Then

$$\{B_1(x_1, \ldots, x_{i_\delta-1}, m + \frac{1}{2}, 0, \ldots) | m \in \mathbb{N}, \ x_k \in \{0\} \cup \{p + \frac{1}{2}, -q - \frac{1}{2} | p, q \in \mathbb{N}\} \quad \text{for each } k \in \{1, \ldots, i_\delta - 1\}\}$$

$$= \bigcup_{s \in U(2, i_\delta - 1)} \{B_1(x_1, \ldots, x_{i_\delta-1}, m + \frac{1}{2}, 0, \ldots) \mid m \in \mathbb{N}, \ x_k \in \{p + \frac{1}{2}, -q - \frac{1}{2} | p, q \in \mathbb{N}\} \quad \text{for each } k \in c_s \text{ and } x_k = 0 \text{ for } k \notin c_s\}.$$

Since $U(2, i_\delta - 1)$ is finite and $\mathcal{S}_\delta$ is infinite, there exists $s_0 \in U(2, i_\delta - 1)$ and an infinite family $\mathcal{A}_\delta \subset \mathcal{S}_\delta$ such that

$$\mathcal{A}_\delta = \{B_1(x^n_1, \ldots, x^n_{i_\delta-1}, m_n + \frac{1}{2}, 0, \ldots) \mid n \in \mathbb{N}, \ x^n_k = 0 \text{ iff } s_0(k) = 0, \ x^n_k \in \{p + \frac{1}{2}, -q - \frac{1}{2} | p, q \in \mathbb{N}\} \text{ iff } s_0(k) \neq 0\}.$$

For every $k \in c_{s_0}$ let

$$S_k = \{x^n_k \mid \text{some } B_1(y_1, \ldots, y_{i_\delta-1}, m + \frac{1}{2}, 0, \ldots) \in \mathcal{A}_\delta \text{ with } y_k = x^n_k\}$$

and let

$$S_{i_\delta} = \{m + \frac{1}{2} | m \in \mathbb{N} \text{ and some } B_1(y_1, \ldots, y_{i_\delta-1}, m + \frac{1}{2}, 0, \ldots) \in \mathcal{A}_\delta\}.$$

Since $\text{card} \mathcal{A}_\delta \leq (\prod_{k \in c_{s_0}} \text{card} S_k) \cdot \text{card} S_{i_\delta}$ and $\mathcal{A}_\delta$ is infinite, there exists $k_0 \in c_{s_0}$ such that $S_{k_0}$ is infinite or $S_{i_\delta}$ is infinite.

Suppose $S_{i_\delta}$ infinite. Then for each $n \in \mathbb{N}$ there is $(w^n_k)_{k \in \mathbb{N}} \in c_0$ such that

$$(w^n_k)_{k \in \mathbb{N}} \in B_\delta((t_k)_{k \in \mathbb{N}}) \cap B_1(x^n_1, \ldots, x^n_{i_\delta-1}, m_n + \frac{1}{2}, 0, \ldots).$$

Thus, from $|w^n_k - t_k| < \delta$ and $|m_n + \frac{1}{2} - w^n_k| < 1$ for every $n \in \mathbb{N}$ it follows that $m_n + \frac{1}{2} \in (t_k - \frac{3}{2}, t_k + \frac{3}{2})$ for every $n \in \mathbb{N}$. This contradicts the assumption that $S_{i_\delta}$ is infinite. Analogously if $S_{k_0}$ is infinite for some $k_0 \in \{1, \ldots, i_\delta - 1\}$.

Whence, from the assumption that $\mathcal{S}_\delta$ is not locally finite in $c_0$ we have an infinite subfamily $\mathcal{F}_\delta$ of $\mathcal{S}_\delta$ which allow us to define a nonvoid set $C_\delta$ which is neither finite nor infinite.

In the covering $\mathcal{U}$ we can attain the constant radius of the balls to be an arbitrary number $r > 0$, because $rB_1(a) = B_r(ra)$ and the homotheties in a Banach space are homeomorphisms.

**Proposition 1.** Let $E$ be a normed space, and $F$ be a subspace of $E$. If $E$ has a locally finite covering by bounded, convex sets, then also there is a locally finite covering of $F$ by bounded, convex sets.

**Proof.** Let $\mathcal{U}$ be a locally finite covering of $E$ by bounded convex sets. Then $\mathcal{V} = \{U \cap F | U \in \mathcal{U}, U \cap F \neq \emptyset\}$ is also a locally finite covering of $F$ by bounded, convex sets.
REMARK. From the last proposition, Theorem 1, and the Corson theorem it follows that \( c_0 \) does not have reflexive, infinite-dimensional Banach subspaces. (This is known; see [1, p. 194].)

**Theorem 2.** Let \( E \) be an infinite-dimensional normed space.

1. If there exists a topology \( T \) on \( E \) such that \((E, T)\) is a \( T_2 \) topological vector space and \( \overline{B}_1(0) \) is compact in \((E, T)\), then there is no covering of \( E \), locally finite in the topology defined by the norm, formed by balls.

2. If there exists a topology \( T \) on \( E \) such that the closed balls of \( E \) are closed in \((E, T)\) and \( \overline{B}_1(0) \) is compact in \((E, T)\), then there is no covering of \( E \), locally finite in the topology defined by the norm, formed by balls.

3. If there exists a topology \( T \) on \( E \) such that \((E, T)\) is \( T_2 \) and for each open, bounded, convex subset of \( E \) its closure in the topology defined by the norm, is compact in \((E, T)\), then there is no covering of \( E \), locally finite in the topology defined by the norm, formed by bounded, convex sets.

(REMARK. (3) generalizes the Corson theorem.)

**Proof.** It is analogous to the proof of the Corson theorem [2].

**Corollary.** If an infinite-dimensional Banach space \( B \) is conjugate, then there is no locally finite covering of \( B \) by balls.

**Proof.** It follows from (1) of Theorem 2 and from the Dixmier-Goldberg-Ruston Theorem [6].

**Remark.** Theorem 1 contradicts A. Pelczyński’s note [9]: “It verifies the result analogous to Corson’s theorem for any Banach space in which there is a weakly closed cone, every closed convex and bounded part of which is weakly compact. Such a cone exists in every Banach space in which there is a weakly convergent sequence which is not norm-convergent.” Nevertheless, this idea allows us to give a new generalization of the Corson theorem.

**Theorem 3.** Let \( E \) be an infinite-dimensional normed space. If there exists a topology \( T \) on \( E \) such that \((E, T)\) is \( T_2 \) and there is a cone \( K \) in \( E \), with nonvoid interior in \( E \) and such that for every open bounded convex subset of \( E \) its closure in the topology defined by the norm on \( E \), relativized to \( K \), is compact in \((E, T)\), then there is no covering of \( E \), locally finite in the topology defined by the norm, formed by bounded, convex sets.

**Proof.** The result follows by an argument analogous to the proof of the Corson theorem.

**Open Problems.**

**Problem 1.** Give a characterization of those infinite-dimensional normed spaces such that the open basis formed by all open balls is coarse.

**Problem 2.** Give an intrinsic description of those infinite-dimensional normed spaces \( E \) which have a topology \( T \) such that \((E, T)\) is a \( T_2 \) topological vector space and \( \overline{B}_1(0) \) is compact in \((E, T)\).

**Problem 3.** Give a characterization of those infinite-dimensional normed spaces such that the open basis formed by all open, bounded, convex subsets is coarse.

**Problem 4.** Does the Banach space \( c_0(A) \), for an arbitrary index set \( A \), admit a locally finite covering by open balls?

(Problems 4 and 5 were raised by the referee.)
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REFERENCES


DEPARTAMENTO DE GEOMETRIA Y TOPOLOGIA, FACULTAD DE MATEMATICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, 28040 MADRID, SPAIN