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TOTAL PARACOMPACTNESS AND BANACH SPACES

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ABSTRACT. In this paper, we study some problems related to the Corson theorem. In particular we prove that c_0 does not fulfil such a theorem; hence this theorem is not valid for all infinite-dimensional Banach spaces. We give also generalizations of Corson's theorem for some infinite-dimensional normed spaces.

A topological space is said to be totally paracompact [**5**] if every open basis contains a locally finite covering. It is known [**4**] that "every totally paracompact complete metric space is **C**-scattered" and "every σ-locally compact paracompact space is totally paracompact". Then, every Banach space is totally paracompact if and only if it is finite dimensional. Thus, every infinite-dimensional Banach space necessarily has an open basis which contains no locally finite covering (i.e. a coarse open basis [**3**]). We raise the question of how to give an intrinsic description of those infinite-dimensional Banach spaces such that the open basis formed by all open balls is coarse. Corson's theorem is related with this question:

THE CORSON THEOREM [**2**]. *For any covering $\mathscr{U}$ of a reflexive, infinite-dimensional Banach space B, where $\mathscr{U}$ is formed by bounded, convex sets, there is a point x in B such that each neighborhood of x meets infinitely many elements of $\mathscr{U}$; i.e., $\mathscr{U}$ is not locally finite.*

In this paper, we study some problems related to the Corson theorem. In particular, we shall prove that c_0 does not fulfil such a theorem, hence this theorem is not valid for all infinite-dimensional Banach spaces. We shall give a generalization of the Corson theorem for some infinite-dimensional normed spaces.

THEOREM 1. *For every $r > 0$, there exists an open covering of c_0, which is locally finite and is formed by balls of radius r.*

PROOF. Let $\mathscr{U} = \{B_1(0)\} \cup \mathscr{U}_+ \cup \mathscr{U}_-$, where

$$\mathscr{U}_+ = \{B_1(x_1, \dots, x_{i-1}, n + \tfrac{1}{2}, 0, \dots) | i \in \mathbf{N},\ n \in \mathbf{N},\ x_k \in \{0\} \cup \{p + \tfrac{1}{2}, -q - \tfrac{1}{2} | p, q \in \mathbf{N}\},\ k \in \{1, \dots, i-1\}\},$$

$$\mathscr{U}_- = \{B_1(y_1, \dots, y_{j-1}, -m - \tfrac{1}{2}, 0, \dots) | j \in \mathbf{N},\ m \in \mathbf{N},\ y_i \in \{0\} \cup \{p + \tfrac{1}{2}, -q - \tfrac{1}{2} | p, q \in \mathbf{N}\},\ i \in \{1, \dots, j-1\}\}.$$

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We shall prove that

(1) $\mathscr{U}$ covers c_0.

(2) $\mathscr{U}$ is locally finite in c_0.

(1) For each $(x_n)_{n\in\mathbf{N}} \in c_0$ there is the first natural number n_0 such that $|x_n| < 1$ for every $n \geq n_0$.

If $n_0 = 1$, we have $|x_n| < 1$ for every $n \geq 1$. Then $\|(x_n)_{n\in\mathbf{N}}\|_\infty < 1$ and $(x_n)_{n\in\mathbf{N}} \in B_1(0) \in \mathscr{U}$.

If $n_0 > 1$, we have that $|x_{n_0-1}| \geq 1$ and

there is $m = E[x_{n_0-1}] \in \mathbf{N}$ such that $|m + \frac{1}{2} - x_{n_0-1}| \leq \frac{1}{2} < 1$ for $x_{n_0-1} \geq 1$;

there is $m = E[-x_{n_0-1}] \in \mathbf{N}$ such that $|-m - \frac{1}{2} - x_{n_0-1}| \leq \frac{1}{2} < 1$ for $x_{n_0-1} \leq -1$.

(We write $E[x]$ for the "integral part of x", the largest integer which does not exceed x.)

Let $(y_n)_{n\in\mathbf{N}} \in c_0$ be such that $y_n = 0$ for every $n \geq n_0$,

$$y_{n_0-1} = \begin{cases} m + \frac{1}{2} & \text{for } x_{n_0-1} \geq 1, \\ -m - \frac{1}{2} & \text{for } x_{n_0-1} \leq -1, \end{cases} \qquad \text{for each } n < n_0 - 1,$$

$$y_n = \begin{cases} 0 & \text{for } |x_n| < 1, \\ E[x_n] + \frac{1}{2} & \text{for } x_n \geq 1, \\ -E[-x_n] - \frac{1}{2} & \text{for } x_n \leq -1. \end{cases}$$

Then $\|(x_n)_{n\in\mathbf{N}} - (y_n)_{n\in\mathbf{N}}\|_\infty < 1$ and $(x_n)_{n\in\mathbf{N}} \in B_1((y_n)_{n\in\mathbf{N}}) \in \mathscr{U}_+ \cup \mathscr{U}_- \subset \mathscr{U}$.

(2) We shall prove that $\mathscr{U}_+$ is locally finite in c_0 (for $\mathscr{U}_-$ it is analogous) and clearly we shall have that $\mathscr{U}$ is locally finite in c_0.

If $\mathscr{U}_+$ is not locally finite in c_0, there exists $(t_k)_{k\in\mathbf{N}} \in c_0$ such that for each $\delta > 0$ there is an infinite family $\mathscr{F}_\delta$, which is contained in $\mathscr{U}_+$ such that $B_\delta((t_k)_{k\in\mathbf{N}}) \cap B \neq \varnothing$ for every $B \in \mathscr{F}_\delta$; then the set

$$\begin{aligned} C_\delta = \{i \in \mathbf{N} | &\text{there is } m \in \mathbf{N}, \text{there are } x_k \in \{0\} \cup \{p + \tfrac{1}{2}, -q - \tfrac{1}{2} | p, q \in \mathbf{N}\} \\ &\text{for each } k \in \{1, \dots, i-1\} \\ &\text{such that } B_1(x_1, \dots, x_{i-1}, m + \tfrac{1}{2}, 0, \dots) \in \mathscr{F}_\delta\} \end{aligned}$$

is nonvoid. Let $0 < \delta < \frac{1}{2}$ in the following argument. C_δ can be infinite or finite.

If C_δ is infinite, then $C_\delta \supset \{i_n | n \in \mathbf{N}\}$ such that $i_n < i_{n+1}$ for every $n \in \mathbf{N}$. Then there exist integers such that

$$\{B_1(x_1^n, \dots, x_{i_n-1}^n, m_n + \tfrac{1}{2}, 0, \dots) | n \in \mathbf{N}\} \in \mathscr{F}_\delta$$

and for every $n \in \mathbf{N}$, there exists $(z_k^n)_{k\in\mathbf{N}} \in c_0$ such that

$$(z_k^n)_{k\in\mathbf{N}} \in B_\delta((t_k)_{k\in\mathbf{N}}) \cap B_1(x_1^n, \dots, x_{i_n-1}^n, m_n + \tfrac{1}{2}, 0, \dots).$$

Thus, from $|z_{i_n}^n - m_n - \frac{1}{2}| < 1$ and $|z_{i_n}^n - t_{i_n}| < \delta$ for every $n \in \mathbf{N}$ it follows that $m_n - \frac{1}{2} < t_{i_n} + \delta$ for every $n \in \mathbf{N}$. Since $m_n \in \mathbf{N}$ for every $n \in \mathbf{N}$, we have that $t_{i_n} > \frac{1}{2} - \delta$ for every $n \in \mathbf{N}$. Then $(t_k)_{k\in\mathbf{N}}$ is not an element of c_0 and this is a contradiction.

If C_δ is finite, let an $i_\delta \in C_\delta$. Then

$$B_\delta((t_k)_{k\in\mathbf{N}}) \cap B_1(x_1, \dots, x_{i_\delta-1}, m + \tfrac{1}{2}, 0, \dots) \neq \varnothing$$

for infinites $(x_1, \dots, x_{i_\delta-1}, m + \frac{1}{2})$, where $x_k \in \{0\} \cup \{p + \frac{1}{2}, -q - \frac{1}{2} | p, q \in \mathbf{N}\}$ for each $k \in \{1, \dots, i_\delta - 1\}$ and $m \in \mathbf{N}$. Then, there exists an infinite family $\mathscr{G}_\delta \subset \mathscr{F}_\delta$

formed by balls $B_1(x_1^n, \dots, x_{i_\delta-1}^n, m_n + \frac{1}{2}, 0, \dots)$ which, we suppose, are pairwise different.

Now, for i_δ let $U(2, i_\delta - 1)$ be the set of all $(i_\delta - 1)$-samples with repetitions of the elements 0 and 1.

For each $s \in U(2, i_\delta - 1)$, let $c_s = \{k \in \{1, \dots, i_\delta - 1\} | s(k) \neq 0\}$.

Then

$$\{B_1(x_1, \dots, x_{i_\delta-1}, m + \tfrac{1}{2}, 0, \dots) | m \in \mathbf{N},\ x_k \in \{0\} \cup \{p + \tfrac{1}{2}, -q - \tfrac{1}{2} | p, q \in \mathbf{N}\} \text{ for each } k \in \{1, \dots, i_\delta - 1\}\}$$
$$= \bigcup_{s \in U(2, i_\delta - 1)} \{B_1(x_1, \dots, x_{i_\delta-1}, m + \tfrac{1}{2}, 0, \dots) | m \in \mathbf{N},\ x_k \in \{p + \tfrac{1}{2}, -q - \tfrac{1}{2} | p, q \in \mathbf{N}\} \text{ for each } k \in c_s \text{ and } x_k = 0 \text{ for } k \notin c_s\}.$$

Since $U(2, i_\delta - 1)$ is finite and $\mathscr{G}_\delta$ is infinite, there exists $s_0 \in U(2, i_\delta - 1)$ and an infinite family $\mathscr{A}_\delta \subset \mathscr{G}_\delta$ such that

$$\mathscr{A}_\delta = \{B_1(x_1^n, \dots, x_{i_\delta-1}^n, m_n + \tfrac{1}{2}, 0, \dots) | n \in \mathbf{N},\ x_k^n = 0 \text{ iff } s_0(k) = 0,\ x_k^n \in \{p + \tfrac{1}{2}, -q - \tfrac{1}{2} | p, q \in \mathbf{N}\} \text{ iff } s_0(k) \neq 0\}.$$

For every $k \in c_{s_0}$ let

$$S_k = \{x_k^n | \text{some } B_1(y_1, \dots, y_{i_\delta-1}, m + \tfrac{1}{2}, 0, \dots) \in \mathscr{A}_\delta \text{ with } y_k = x_k^n\}$$

and let

$$S_{i_\delta} = \{m + \tfrac{1}{2} | m \in \mathbf{N} \text{ and some } B_1(y_1, \dots, y_{i_\delta-1}, m + \tfrac{1}{2}, 0, \dots) \in \mathscr{A}_\delta\}.$$

Since $\operatorname{card} \mathscr{A}_\delta \leq (\prod_{k \in c_{s_0}} \operatorname{card} S_k) \cdot \operatorname{card} S_{i_\delta}$ and $\mathscr{A}_\delta$ is infinite, there exists $k_0 \in c_{s_0}$ such that S_{k_0} is infinite or S_{i_δ} is infinite.

Suppose S_{i_δ} infinite. Then for each $n \in \mathbf{N}$ there is $(w_k^n)_{k \in \mathbf{N}} \in c_0$ such that

$$(w_k^n)_{k \in \mathbf{N}} \in B_\delta((t_k)_{k \in \mathbf{N}}) \cap B_1(x_1^n, \dots, x_{i_\delta-1}^n, m_n + \tfrac{1}{2}, 0, \dots).$$

Thus, from $|w_{i_\delta}^n - t_{i_\delta}| < \delta$ and $|m_n + \frac{1}{2} - w_{i_\delta}^n| < 1$ for every $n \in \mathbf{N}$ it follows that $m_n + \frac{1}{2} \in (t_{i_\delta} - \frac{3}{2}, t_{i_\delta} + \frac{3}{2})$ for every $n \in \mathbf{N}$. This contradicts the assumption that S_{i_δ} is infinite. Analogously if S_{k_0} is infinite for some $k_0 \in \{1, \dots, i_\delta - 1\}$.

Whence, from the assumption that $\mathscr{U}_+$ is not locally finite in c_0 we have an infinite subfamily $\mathscr{F}_\delta$ of $\mathscr{U}_+$ which allow us to define a nonvoid set C_δ which is neither finite nor infinite.

In the covering $\mathscr{U}$ we can attain the constant radius of the balls to be an arbitrary number $r > 0$, because $rB_1(a) = B_r(ra)$ and the homotheties in a Banach space are homeomorphisms.

PROPOSITION 1. *Let E be a normed space, and F be a subspace of E. If E has a locally finite covering by bounded, convex sets, then also there is a locally finite covering of F by bounded, convex sets.*

PROOF. Let $\mathscr{U}$ be a locally finite covering of E by bounded convex sets. Then $\mathscr{V} = \{U \cap F / U \in \mathscr{U},\ U \cap F \neq \varnothing\}$ is also a locally finite covering of F by bounded, convex sets.

REMARK. From the last proposition, Theorem 1, and the Corson theorem it follows that c_0 does not have reflexive, infinite-dimensional Banach subspaces. (This is known; see [**1**, p. 194].)

THEOREM 2. *Let E be an infinite-dimensional normed space.*

(1) *If there exists a topology T on E such that (E,T) is a T_2 topological vector space and $\overline{B}_1(0)$ is compact in (E,T), then there is no covering of E, locally finite in the topology defined by the norm, formed by balls.*

(2) *If there exists a topology T on E such that the closed balls of E are closed in (E,T) and $\overline{B}_1(0)$ is compact in (E,T), then there is no covering of E, locally finite in the topology defined by the norm, formed by balls.*

(3) *If there exists a topology T on E such that (E,T) is T_2 and for each open, bounded, convex subset of E its closure in the topology defined by the norm, is compact in (E,T), then there is no covering of E, locally finite in the topology defined by the norm, formed by bounded, convex sets.*

(REMARK. (3) generalizes the Corson theorem.)

PROOF. It is analogous to the proof of the Corson theorem [**2**].

COROLLARY. *If an infinite-dimensional Banach space B is conjugate, then there is no locally finite covering of B by balls.*

PROOF. It follows from (1) of Theorem 2 and from the Dixmier-Goldberg-Ruston Theorem [**6**].

REMARK. Theorem 1 contradicts A. Pełczyński's note [**9**]: "It verifies the result analogous to Corson's theorem for any Banach space in which there is a weakly closed cone, every closed convex and bounded part of which is weakly compact. Such a cone exists in every Banach space in which there is a weakly convergent sequence which is not norm-convergent." Nevertheless, this idea allows us to give a new generalization of the Corson theorem.

THEOREM 3. *Let E be an infinite-dimensional normed space. If there exists a topology T on E such that (E,T) is T_2 and there is a cone K in E, with nonvoid interior in E and such that for every open bounded convex subset of E its closure in the topology defined by the norm on E, relativized to K, is compact in (E,T), then there is no covering of E, locally finite in the topology defined by the norm, formed by bounded, convex sets.*

PROOF. The result follows by an argument analogous to the proof of the Corson theorem.

OPEN PROBLEMS.

Problem 1. Give a characterization of those infinite-dimensional normed spaces such that the open basis formed by all open balls is coarse.

Problem 2. Give an intrinsic description of those infinite-dimensional normed spaces E which have a topology T such that (E,T) is a T_2 topological vector space and $\overline{B}_1(0)$ is compact in (E,T).

Problem 3. Give a characterization of those infinite-dimensional normed spaces such that the open basis formed by all open, bounded, convex subsets is coarse.

Problem 4. Does the Banach space $c_0(A)$, for an arbitrary index set A, admit a locally finite covering by open balls?

Problem 5. Does the James Hagler space [J. Hagler, *A counterexample to several questions about Banach spaces*, Studia Math. **60** (1977), 289–308] admit a locally finite covering by open balls?

(Problems 4 and 5 were raised by the referee.)

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