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THE FABER TRANSFORM AND ANALYTIC CONTINUATION

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ABSTRACT. Let $\Omega \subseteq C$ be a bounded, simply connected domain, and let $\{\Phi_n(w)\}_{n=0}^{\infty}$ be the Faber polynomials associated with Ω. Given $f(z) = \sum_{k=0}^{\infty} c_k z^k$ analytic in $\Delta(0,1)$ we consider the function

$$F(w) = \sum_{k=0}^{\infty} c_k \Phi_k(w).$$

We show that with proper restrictions on $\partial\Omega$, the existence of an analytic continuation of f across a subarc of $C(0,1)$ implies the existence of an analytic continuation of F across a subarc of $\partial\Omega$. Some converse results are also established.

1. Introduction. Let $\Omega \subseteq \mathbf{C}$ be a bounded, simply connected domain for which $\mathbf{C} \setminus \overline{\Omega}$ is connected. We use $g(\xi)$ to denote the unique function that is analytic and univalent on $\{|\xi| > 1\}$, maps $\{|\xi| > 1\}$ onto the exterior of Ω, and has expansion

$$g(\xi) = b_{-1}\xi + b_0 + b_1/\xi + b_2/\xi^2 + \cdots \qquad (b_{-1} > 0) \tag{1}$$

in a neighborhood of ∞. The Faber polynomials associated with Ω (or g) are the polynomials $\{\Phi_n(w)\}_{n=0}^{\infty}$ determined by the following generating function relationship [*P*]:

$$\frac{g'(\xi)}{g(\xi) - w} = \sum_{k=0}^{\infty} \Phi_k(w)\xi^{-k-1}.$$

It can be shown that with proper restrictions on ∂D (see for example [**S**]) any function $F(w)$ analytic in Ω has a unique "Faber expansion"

$$\sum_{k=0}^{\infty} c_k \Phi_k(w) \tag{2}$$

that converges to $F(w)$ uniformly on compact subsets of Ω. Conversely (again with appropriate restriction on $\partial\Omega$) any expression of the form (2) converges uniformly on compact subsets of Ω provided $\limsup_{k\to\infty} \sqrt[k]{|c_k|} \le 1$.

In view of the last statement, we see that if

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \tag{3}$$

is analytic on the unit disk $\Delta(0,1)$, then the function $F(w) = \sum_{k=0}^{\infty} c_k \Phi_k(w)$ might be analytic on Ω. The operator $\mathscr{F}$ that takes a function (3) analytic on $\Delta(0,1)$ and maps it to the (formal) Faber series $\sum_{k=0}^{\infty} c_k \Phi_k(w)$ is called the Faber transform.

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In this paper we address some aspects of the following question: if $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is analytic on $\Delta(0,1)$ and has a certain property P, then to what extent is this property "inherited" by $F(w) = \mathscr{F}(f(z))(w) = \sum_{k=0}^{\infty} c_k \Phi_k(w)$? For example, it is clear that if $f(z)$ is a polynomial, then so is $\mathscr{F}(f(z))(w)$. Ellacott (and Gaier) [**E**] have shown that if $f(z)$ is a rational function then the same is true of $\mathscr{F}(f(z))(w)\ldots$ no restrictions on $\partial\Omega$ are required. In this paper we consider the question of analytic continuation. We show that with proper restriction on $\partial\Omega$ and/or on $f(z)$, $\mathscr{F}(f(z))(w)$ has analytic continuation properties similar to those of $f(z)$.

In the remainder of the paper, $g(\xi)$ will be as defined in (1). We will also assume $b_{-1} = 1\ldots$ this assumption simply results in a change of scale and has no affect on the results.

2. $\partial\Omega$ analytic. As one would expect, the easiest case to consider is that in which $\partial\Omega$ is analytic.

THEOREM 1. *Let $\partial\Omega$ be analytic and let $J \subseteq \partial\Omega$ be a subarc of $\partial\Omega$. For a given $f(z)$ analytic on $\Delta(0,1)$, the function $F(w) = \mathscr{F}(f(z))(w)$ is analytic on Ω and has an analytic continuation across J if and only if $f(z)$ has an analytic continuation across $g^{-1}(J)$.*

PROOF. The fact that $F(w)$ is analytic on Ω is well known (see [**S**]). Since $\partial\Omega$ is analytic, $g(\xi)$ can be analytically and univalently continued to some domain $\{|\xi| > r_0\}$ for some $r_0 \in (0,1)$. Hence if $f(z) = \sum_{k=0}^{\infty} c_k z^k$ on $\Delta(0,1)$, then for $r_0 < |\xi| < 1$ we have

$$\begin{aligned} F(g(\xi)) &= \sum_{k=0}^{\infty} c_k \Phi_k(g(\xi)) \\ &= \sum_{k=0}^{\infty} c_k \left(\xi^k + k \sum_{l=1}^{\infty} b_{kl} \xi^{-l} \right) \qquad (4) \\ &= f(\xi) + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k c_k b_{kl} \xi^{-1} \end{aligned}$$

where the coefficients $\{b_{kl}\}_{k,l=1}^{\infty}$ are the Grunsky coefficients associated with g (see [**P**]). Since $g(\xi)$ is analytic and univalent in $\{|\xi| > r_0\}$, it follows that for any $r \in (r_0, 1)$, we have $|b_{kl}| = o(r^{k+l})$ as $k + l \to \infty$. Thus the last sum in (4) defines a function analytic in $\{|\xi| > r_0\}$. Hence, if $f(\xi)$ can be continued analytically across $g^{-1}(J) \subseteq \{|\xi| = 1\}$, then the same is true of $F(g(\xi))$. Let $h(\xi)$ denote such an analytic continuation. Then for some open neighborhood $N(g^{-1}(J)) \subseteq \{|\xi| > r_0\}$ of $g^{-1}(J)$ we have $h(\xi)$ analytic on $\{r_0 < |\xi| < 1\} \cup N(g^{-1}(J))$ and $h(\xi) \equiv F(g(\xi))$ on $\{r_0 < |\xi| < 1\}$. Define

$$\tilde{F}(w) = \begin{cases} F(w) & (w \in \Omega), \\ h(g^{-1}(w)) & (w \in g(\{r_0 < |\xi| < 1\} \cap N(g^{-1}(J)))). \end{cases}$$

Then $\tilde{F}(w)$ is an analytic continuation of F across J.

Conversely, if $F(w)$ has an analytic continuation across J, then $F(g(\xi))$, defined for $r_0 < |\xi| < 1$, has an analytic continuation across $g^{-1}(J)$. It then follows from (4) that $f(z)$ has an analytic continuation across $g^{-1}(J)$. $\square$

In view of Theorem 1, we see that results concerning analytic continuations of functions analytic on $\Delta(0,1)$ have counter parts for functions defined by Faber expansions. For example, we have a "Faber-Hadamard" gap theorem.

COROLLARY. *Let $\partial\Omega$ be analytic and let*

$$F(w)=\sum_{k=1}^{\infty} c_{n_k}\Phi_{n_k}(w)$$

be analytic in Ω, but not on any neighborhood of $\overline{\Omega}$. Suppose there is a $\lambda>1$ such that the integer sequence $\{n_k\}_{k=1}^{\infty}$ satisfies $n_{k+1}/n_k\geq\lambda$ $(k\geq k_0)$. Then $\partial\Omega$ is the natural boundary of F.

3. Nonanalytic $\partial\Omega$. If $\partial\Omega$ is not analytic, then there is no guarantee that $F(w)=\mathscr{F}(f(z))(w)$ defines a function analytic on Ω. Something can be said, however, in the case where $\partial\Omega$ is a curve of bounded rotation and $f(z)\in A(\overline{\Delta(0,1)})=\{f(z)\colon f \text{ analytic on } \Delta(0,1), \text{ continuous on } \overline{\Delta(0,1)}\}$. Even if ∂D is of bounded rotation and $f\in A(\overline{\Delta(0,1)})$ with

$$f(z)=\sum_{k=0}^{\infty} c_k z^k,$$

it is possible that $\sum_{k=0}^{\infty} c_k\Phi_k(w)$ does not define a function in $A(\overline{\Omega})$. However if, in this case, we define

$$F(w)=\mathscr{F}(f(z))(w)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(g^{-1}(\xi))}{\xi-w}\,d\xi \qquad (w\in\Omega), \tag{5}$$

then $F\in A(\overline{\Omega})$ and has Faber coefficients $\{c_k\}_{k=0}^{\infty}$ agreeing with the McLauren coefficients of f (see [**G**]). In this case we use the integral expression

$$c_k=\frac{1}{2\pi i}\int_{|z|=1}\frac{F(g(z))}{z^{n+1}}\,dz$$

for the coefficients. We remark that it is easily checked that (5) agrees with the "coefficient transplant" description of $\mathscr{F}$ in the case when $\partial\Omega$ is analytic.

Thus if $\partial\Omega$ is of bounded rotation, then (5) defines a linear mapping

$$\mathscr{F}\colon A(\overline{\Delta(0,1)})\to A(\overline{\Omega}).$$

In fact, $\mathscr{F}$ is a continuous operator with respect to the supremum norms on $A(\overline{\Delta(0,1)})$ and $A(\overline{\Omega})$, with $\|\mathscr{F}\|\leq(1+2V/\pi)$, where V is the total rotation of $\partial\Omega$. (See [**G**].)

THEOREM 2. *Let $\partial\Omega$ be of bounded rotation and let $f\in A(\overline{\Delta(0,1)})$. If $f(z)$ has an analytic continuation across the arc $I\subseteq C(0,1)$, then $F(w)=\mathscr{F}(f(z))(w)$ has an analytic continuation across $g(I)$.*

PROOF. We first assume I is a closed subarc of the unit circle, i.e. $I=\{e^{i\theta}\colon\alpha\leq\theta\leq\beta\}$. Let $h(z)$ be an analytic continuation of f to $\Delta(0,1)\cup N(I)$ where $N(I)$ is some open neighborhood of I. Since I is closed, we may assume h is defined and continuous on $\overline{\Delta(0,1)\cup N(I)}$ and that $\mathbf{C}\setminus\overline{\Delta(0,1)\cup N(I)}$ is connected. By Mergelyan's Theorem [**R**] there is a sequence $\{P_n(z)\}_{n=1}^{\infty}$ of polynomials such that

$P_n(z) \to h(z)$ uniformly on $\overline{\Delta(0,1) \cup N(I)}$. Now consider the sequence of polynomials

$$Q_n(w) = \mathscr{F}(P_n(z))(w) \qquad (n = 1, 2, 3, \dots).$$

For $w \in \overline{\Omega}$ we have

$$\begin{aligned} |Q_n(w) - F(w)| &= |\mathscr{F}(P_n(z))(w) - \mathscr{F}(f(z))(w)| \\ &\le \|\mathscr{F}\| \left(\sup_{|z|\le 1} |P_n(z) - f(z)| \right) \\ &= \|\mathscr{F}\| \left(\sup_{|z|\le 1} |P_n(z) - h(z)| \right) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{6}$$

As equation (6) shows, $\{Q_n(w)\}_{n=1}^{\infty}$ converges uniformly to F on $\overline{\Omega}$. We claim that $\{Q_n(w)\}_{n=1}^{\infty}$ also converges uniformly on $g(\overline{N(I)} \cap \{|\xi| \ge 1\})$.

Given $w \in g(\overline{N(I)} \cap \{|z| > 1\})$, find $\xi \in \overline{N(I)} \cap \{|z| > 1\}$ with $w = g(\xi)$. Writing

$$P_n(\xi) = \sum_{k=0}^{m_n} c_k^{(n)} \xi^k,$$

we have

$$\begin{aligned} Q_n(w) &= \sum_{k=0}^{m_n} c_k^{(n)} \Phi_k(w) = \sum_{k=0}^{m_n} c_k^{(n)} \Phi_k(g(\xi)) \\ &= \sum_{k=0}^{m_n} c_k^{(n)} \xi^n + \sum_{k=0}^{m_n} \left(k c_k^{(n)} \sum_{l=1}^{\infty} b_{kl} \xi^{-l} \right) \\ &= P_n(\xi) + \sum_{l=1}^{\infty} \left(\sum_{k=0}^{m_n} k c_k^{(n)} b_{kl} \right) \xi^{-l}. \end{aligned}$$

Now

$$R_n(\xi) = Q_n(g(\xi)) - P_n(\xi) = \sum_{l=1}^{\infty} \left(\sum_{k=0}^{m_n} k c_k^{(n)} b_{kl} \right) \xi^{-l}$$

is analytic in $|\xi| > 1$ and vanishes at ∞. Furthermore, for a given real θ,

$$\lim_{\substack{\xi \to e^{i\theta} \\ |\xi| > 1}} R_n(\xi) = Q_n(g(e^{i\theta})) - P_n(e^{i\theta}).$$

Since $\{P_n\}$ is uniformly Cauchy on $\overline{\Delta(0,1)}$ and $\{Q_n(w)\}$ is uniformly Cauchy on $\overline{\Omega}$, it follows from the Maximum Modulus Theorem that $\{R_n(\xi)\}$ is uniformly Cauchy on $\{|\xi| \ge 1\}$. Now for $w = g(\xi) \in g(\overline{N(I)} \cap \{|z| \ge 1\})$, we have

$$\begin{aligned} |Q_n(w) - Q_m(w)| &= |Q_n(g(\xi)) - Q_m(g(\xi))| \\ &\le |P_n(\xi) - P_m(\xi)| + |R_n(\xi) - R_m(\xi)|. \end{aligned}$$

Since $\{P_n(\xi)\}$ is uniformly Cauchy on $\overline{\Delta(0,1) \cup N(I)}$ and $\{R_n(\xi)\}$ is uniformly Cauchy on $\{|\xi| \ge 1\}$, it follows that $\{Q_n(w)\}$ is uniformly Cauchy in $g(\overline{N(I)} \cap \{|\xi| \ge 1\})$. Combining with (6) we see that $\{Q_n(w)\}$ is uniformly Cauchy on $\overline{\Omega} \cup g(\overline{N(I)} \cap \{|\xi| \ge 1\}) \dots$ the continuity of g on $\{|\xi| \ge 1\}$ and the analyticity of g on $\{|\xi| > 1\}$ imply this last set is a neighborhood of the (closed) arc $g(I)$. Letting

$$H(w) = \lim_{n\to\infty} Q_n(w)$$

we have a function continuous on $\overline{\Omega} \cup g(\overline{N(I)} \cap \{|\xi| \geq 1\})$ and analytic on the interior of this set. Since this interior contains $g(I)$ and since $H|_{\overline{\Omega}} = F$, H is the desired analytic continuation.

Suppose now the subarc I of the unit circle is not closed. Again let $N(I)$ be an open set containing I with $\mathbf{C} \setminus (\Delta(0,1) \cup N(I))$ connected and suppose $h(z)$ is an analytic continuation of f to $\Delta(0,1) \cup N(I)$. We write $I = \bigcup_{n=1}^{\infty} I_n$ where $\{I_n\}_{n=1}^{\infty}$ is an increasing sequence of closed subarcs of I. We can also find an increasing sequence $\{N(I_n)\}_{n=1}^{\infty}$ of open sets satisfying $I_n \subseteq N(I_n) \subseteq \overline{N(I_n)} \subseteq N(I_{n+1})$ $(n = 1, 2, \dots)$, with each $\mathbf{C} \setminus (\overline{\Delta(0,1)} \cup \overline{N(I)})$ connected and $\bigcup_{n=1}^{\infty} N(I_n) = N(I)$. Then $h_n(z) = h|_{\Delta(0,1)\cup N(I_n)}$ is an analytic continuation of f across I_n, and has a continous extension to $\overline{\Delta(0,1) \cup N(I_n)}$. As shown in the first part of the proof, we can find, for each n, a function $H_n(w)$ continuous on $\overline{\Omega} \cup g(N(I_n) \cap \{|\xi| \geq 1\})$, analytic on the interior of this set and with $H_n|_{\overline{\Omega}} = F$. Since

$$\Omega \cup g(N(I) \cap \{|\xi| \geq 1\}) = \bigcup_{n=1}^{\infty} [\Omega \cup g(N(I_n) \cap \{|\xi| \geq 1\})]$$

as an increasing union, we may define H on $\Omega \cup (g(N(I) \cap \{|\xi| \geq 1\}))$ by

$$H(w) = \begin{cases} F(\omega) & (w \in \Omega), \\ H_n(w) & (w \in g(N(I_n) \cap \{|\xi| \geq 1\})). \end{cases}$$

It is clear that H is well defined and analytic on $\Omega \cup \{g(N(I) \cap \{|\xi| \geq 1\})\}$ and hence is the desired analytic continuation. □

If $\mathscr{F}^{-1}\colon A(\overline{\Omega}) \to A(\overline{\Delta(0,1)})$ is defined and continuous, then the converse of Theorem 2 holds. Unfortunately, the existence of $\mathscr{F}^{-1}$ is not automatic, even if $\partial\Omega$ is of bounded rotation. If $\partial\Omega$ is of bounded rotation, then $\mathscr{F}: A(\overline{\Delta(0,1)}) \to A(\overline{\Omega})$ is a one-to-one mapping, but may not be onto. In fact, given $F \in A(\overline{\Omega})$ ($\partial\Omega$ of bounded rotation), one can assert that $F = \mathscr{F}(f)$ for some $f \in A(\overline{\Delta(0,1)})$ if and only if $h = F \circ g$ and its conjugate, $\tilde{h}$, are both continuous on $C(0,1)$ (see [**G**, p. 53]). In this case $F = \mathscr{F}(f)$ where

$$f(z) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{F(g(\xi))}{\xi - z}\, d\xi.$$

However, if $\mathscr{F}^{-1}\colon A(\overline{\Omega}) \to A(\overline{\Delta(0,1)})$ is defined (i.e. $\mathscr{F}: A(\overline{\Delta(0,1)}) \to A(\overline{\Omega})$ is onto), then $\mathscr{F}^{-1}$ is continuous by the Open Mapping Theorem [**R**].

THEOREM 3. *Let $\partial\Omega$ be of bounded rotation and let $\mathscr{F}: A(\overline{\Delta(0,1)}) \to A(\overline{\Omega})$ be onto. Let $F(w) \in A(\overline{\Omega})$ and J be a subarc of $\partial\Omega$. Then $F(w)$ has an analytic continuation across J if and only if $f(z) = \mathscr{F}^{-1}(F(w))(z)$ has an analytic continuation across $g^{-1}(J)$.*

PROOF. The sufficiency was established in Theorem 2. The proof of the necessity is very similar so we may be somewhat brief. We consider the case in which J is a "closed subarc" of $\partial\Omega\dots$ that is $J = g(\{e^{i\theta}\colon \alpha \leq \theta \leq \beta\})$ for some real α and β.

The cases of "open" or "half-open" J are then taken care of as in the last part of the proof of Theorem 2.

Let $N(J)$ be an open set containing J such that $\mathbf{C} \setminus \overline{\Omega \cup N(J)}$ is connected, $F(w)$ has an analytic continuation, $H(w)$, to $\Omega \cup N(J)$ and H has a continuous extension to $\overline{\Omega \cup N(J)}$.

By Mergelyan's Theorem, there is a sequence $\{Q_n(w)\}$ of polynomials with $Q_n(w) \to H(w)$ uniformly on $\overline{\Omega \cup N(J)}$. Now each $P_n(z) = \mathscr{F}^{-1}(Q_n(w))(z)$ is also a polynomial, and the continuity of $\mathscr{F}^{-1}$ implies that $P_n(z) \to f(z) = \mathscr{F}^{-1}(F(w))(z)$ uniformly on $\overline{\Delta(0,1)}$. We claim that the sequence $\{P_n(z)\}$ is also uniformly convergent on $g^{-1}(\overline{N(J)} \setminus (\mathbf{C} \setminus \Omega))$. To see this write

$$Q_n(w) = \sum_{k=0}^{m_n} c_k^{(n)} \Phi_k(w)$$

as a Faber expansion. Then

$$P_n(z) = \sum_{k=0}^{m_n} c_k^{(n)} z^k.$$

Let $z \in \{|\xi| > 1\}$, $|z|$ large. We can find $w \in \mathbf{C} \setminus \Omega$ ($|w|$ large) with $g^{-1}(z) = w$. Then

$$\begin{aligned} P_n(z) = P_n(g^{-1}(w)) &= \sum_{k=0}^{m_n} c_k^{(n)} (g^{-1}(w))^k \\ &= \sum_{k=0}^{m_n} c_k^{(n)} \left[\Phi_k(w) + \sum_{l=1}^{\infty} d_{kl} w^{-l} \right] \\ &= Q_n(w) + \sum_{l=1}^{\infty} \left(\sum_{k=0}^{m_n} c_k^{(n)} d_{kl} \right) w^{-l} \end{aligned}$$

for some choice of coefficients d_{kl} (see [**G**]). We note that

$$R_n(z) = P_n(z) - Q_n(g(z)) = \sum_{l=1}^{\infty} \left(\sum_{k=0}^{m_n} c_k^{(n)} d_{kl} \right) (g(z))^{-l}$$

(as above, defined for $|z|$ large) can be continued analytically to $\{|z| > 1\}$, and continuously to $\{|z| \geq 1\}$. Furthermore, $R_n(z)$ vanishes at ∞. Since $P_n(z)$ and $Q_n(g(z))$ are both uniformly Cauchy on $C(0,1)$ it follows from the Maximum Modulus Theorem that $\{R_n(z)\}$ is uniformly Cauchy on $\{|z| \geq 1\}$. From this point we may proceed as in the proof of Theorem 2 and assert that

$$h(z) = \lim_{n\to\infty} P_n(z) = \begin{cases} f(z) & (z \in \Delta(0,1)), \\ \lim\limits_{n\to\infty} (R_n(z) + Q_n(g(z))) & (z \in g^{-1}(N(J) \setminus (\mathbf{C} \setminus \Omega))) \end{cases}$$

gives an analytic continuation of f across $g^{-1}(J)$. □

We conclude by noting that Theorem 3 allows us to state a "Faber-Hadamard" gap theorem for $A(\overline{\Omega})$.

COROLLARY. *Suppose $\partial\Omega$ is of bounded rotation and $\mathscr{F} : A(\overline{\Delta(0,1)}) \to A(\overline{\Omega})$ is onto. Let $F(w) \in A(\overline{\Omega})$ have Faber series*

$$F(w) \sim \sum_{k=0}^{\infty} c_{n_k} \Phi_{n_k}(w).$$

If $\limsup_{k\to\infty} |c_{n_k}|^{1/n_k} = 1$ and there is a number $\lambda > 1$ with $n_{k+1}/n_k > \lambda$ $(k > k_0)$, then $\partial\Omega$ is the natural boundary for F.

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