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HORIZONTAL HOLOMORPHIC CURVES IN Sp(n)-FLAG MANIFOLDS

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ABSTRACT. A complete description of horizontal holomorphic curves in Sp(n)-flag manifolds is given. For compact curves a Plücker type integral formula is derived.

0. Introduction. Define the G-flag manifold to be the homogeneous space G/T, where G is a connected compact simple Lie group and T is a maximal torus in G. Also define a partial G-flag manifold to be any homogeneous space of the form G/H, where H is a connected and closed subgroup of maximal rank and G, as in the above. Since H is of maximal rank, replacing T by one of its conjugates if necessary, we assume that $T \subset H$. Then there is the projection $\pi\colon G/T \to G/H$ given by $gT \mapsto gH$. A significance of G-flag manifolds lies in that they are the fundamental building blocks of all compact complex homogeneous spaces (see [**B-H**, **Wan**]).

G/T is naturally a complex manifold (i.e., there is an integrable invariant complex structure on it) and it carries so called the horizontal distribution which is a rank r ($r = \operatorname{rank} G$) holomorphic distribution in the tangent bundle. A holomorphic curve in G/T is said to be horizontal if it is tangent to the horizontal distribution. Horizontal holomorphic curves play a crucial role in studying harmonic maps of surfaces into certain Riemannian homogeneous spaces. For example, Bryant [**Br**, §4] proved that a horizontal holomorphic curve in G/T upon projection down to G/H gives a harmonic map if G/H is a type I inner symmetric space. Note that (irreducible) type I inner symmetric spaces form a small subclass in the set of all partial G-flag manifolds.

Though it has never been explicitly quantized, it is the case that a large class of 2-dimensional harmonic maps into Riemannian homogeneous spaces arises as horizontal holomorphic curves. Let $G = \mathrm{SU}(n+1)$, $T = S(U(1)^{n+1})$ and $H = S(U(1) \times U(n))$. Then there is the following well-known theorem.

THEOREM (CF. [**E-W**, **C-W**]). *Let $g\colon S \to G/H = CP^n$ be a holomorphic curve. Then its "Frenet frame" defines a horizontal holomorphic global lifting of g into G/T. Conversely, given a horizontal holomorphic curve f in G/T the map $\pi \circ f$ gives a holomorphic curve in CP^n. Moreover, a class of weakly conformal harmonic maps into CP^n is obtained by varying the integrable invariant complex structure on G/T.*

The SU(n)-case is exceptional in that any Riemann surface can be locally embedded as a horizontal holomorphic curve in $\mathrm{SU}(n)/T$. The proof of this fact is

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an elementary application of the Cartan-Kähler theory, or one can use the above theorem together with the fact that CP^n is complex.

In this paper we give a complete description of horizontal holomorphic curves in $\mathrm{Sp}(n)$-flag manifolds. It should be noted that in doing so we are at the same time producing a class of harmonic maps into HP^{n-1} which is a partial $\mathrm{Sp}(n)$-flag manifold and a type I inner symmetric space. Compare [**G**].

Let $f\colon S \to \mathrm{Sp}(n)/T$ be a horizontal holomorphic curve from a connected Riemann surface S. Disregarding the branch points of f (which are isolated anyhow) and assigning S the metric induced by f we assume that f is an isometric immersion. Having done so we are able to construct what we call the "Gauss-Frenet map of f" $\phi_f\colon S \to S^{n-1}$, $\phi_f(x) = (a_i(x))$, $\Sigma a_i^2(x) = 1$. ϕ_f in turn completely determines the curve f and the component functions of ϕ_f satisfy a system of generalized Euler-Lagrange equations (10). In the language of exterior differential system these equations are the complete integrability conditions. To put it another way, given functions (a_i) satisfying the equations in (10) one can produce f with $\phi_f = (a_i)$ using integration involving ordinary differential equations only. We also show that the equations in (10) imply exactly one affine relation (11) among the Gaussian curvature of S (with the induced metric) and the Laplacians of the logarithms of (a_i).

In §4 we assume that S is compact and integrate (11). The resulting integral formula gives an equation relating the numbers of zeros of (a_i), the Euler-Poincaré characteristic of S and the area of the immersion.

1. G-flag manifolds. The results in this section are quite standard and are included here mainly to set up the notation we use later. The reader may consult [**B-H**], or [**Y2**] for a detailed exposition. Throughout G is assumed to be connected compact and simple.

Consider the representation of a maximal torus given by $\mathrm{Ad}|_T\colon T \to \mathrm{GL}(\mathfrak{g})$. Impose a bi-invariant metric on G, which exists since G is compact. This gives an $\mathrm{Ad}(G)$-invariant inner product in $\mathfrak{g}$ and relative to an orthonormal basis of this inner product we have $\mathrm{Ad}(T) \subset \mathrm{SO}(2n; R)$. Such a representation decomposes into the direct sum of irreducible representations on its invariant subspaces with the dimensions of invariant subspaces 1 or 2. Write $\mathrm{Ad}|_T = \rho_0 \oplus \rho_1 \oplus \cdots \oplus \rho_n = \mathrm{Ad}|_{V_0} \oplus \mathrm{Ad}|_{V_1} \oplus \cdots \oplus \mathrm{Ad}|_{V_n}$, where $\dim V_i = 2$ for $i > 0$ and $V_0 = \mathfrak{t} =$ the direct sum of one-dimensional invariant subspaces. Let $\{E_\alpha\}$ denote the above orthonormal basis for $\mathfrak{g}$ so that $\{E_{2i-1}, E_{2i}\}$ span V_i for $i > 0$. Thus with respect to $\{E_{2i-1}, E_{2i}\}$ we have

$$\rho_i\colon T \to \mathrm{SO}(2; R), \quad t \mapsto \begin{pmatrix} \cos\theta_i(t), & -\sin\theta_i(t) \\ \sin\theta_i(t), & \cos\theta_i(t) \end{pmatrix} = e^{i\theta_i(t)}$$

(using the identification $\mathrm{SO}(2) = U(1)$) for $i > 0$.

DEFINITION. $\Delta = \{\pm\theta_i\}$ are called the roots of G. (Sometimes $\{\pm\theta_{i^*e}\}$ are also called the roots of G. We do not distinguish between the two.)

PROPOSITION. *There are exactly 2^n ($n = \frac{1}{2}\dim G/T$) many invariant almost complex structures on G/T.*

PROOF. An invariant almost complex structure on G/T is identified, by restriction, with an $\mathrm{Ad}(T)$-invariant complex structure on $\mathfrak{m} = \mathfrak{t}^\perp = V_1 \oplus \cdots \oplus V_n$

(remember that $\mathfrak{m}$ is identified with $T_T(G/T)$ via $\pi_{*e}, \pi\colon G \to G/T$). Now any $\mathrm{Ad}(T)$-invariant complex structure on $\mathfrak{m}$ must leave the invariant subspaces V_i fixed and on each V_i there are exactly two invariant complex structures $\pm J_i$ represented by the matrices $\pm\left(\begin{smallmatrix} 0, & -1 \\ 1, & 0 \end{smallmatrix}\right)$ written relative to the orthonormal basis $\{E_{2i-1}, E_{2i}\}$ of V_i. Q.E.D.

Let J be an invariant almost complex structure on G/T. Using the above proof it makes sense to write $J = \varepsilon_1 J_1 \oplus \cdots \oplus \varepsilon_n J_n$, $\varepsilon_i = \pm 1$.

Recall that $\Delta_+ = \{\varepsilon_i \theta_i\}$ forms a system of positive roots if and only if the set $WC = \{x \in \mathfrak{t}\colon \varepsilon_i\theta_i(x) > 0\}$ is nonempty. WC when nonempty is the fundamental Weyl chamber associated with Δ_+.

PROPOSITION. *$J = \varepsilon_1 J_1 \oplus \cdots \oplus \varepsilon_n J_n$ represents an integrable invariant almost complex structure on G/T if and only if $\{\varepsilon_i\theta_i\}$ forms a system of positive roots.*

We skip the proof. (A proof can be found in [**B-H**, pp. 503–504].)

DEFINITION. The Weyl group of G, denoted by $W(G)$, is the group of automorphisms of T which are the restrictions of inner automorphisms of G. So, $W(G)$ is naturally isomorphic to $N(T)/T$.

PROPOSITION. *There are exactly $|W(G)|$ many integrable invariant almost complex structures on G/T.*

For the proof see [**B-H**, p. 504].

Although all of the integrable invariant complex structures on G/T are equivalent under Diffeo(G/T) there are advantages to keeping them distinct. Indeed, this view is exploited in a subtle way to give a classification of minimal spheres in complex projective spaces in the work [**C-W**].

Let Ω denote the Maurer-Cartan form of G. It is a $\mathfrak{g}$-valued left-invariant 1-form on G and is given by $\Omega_g(X) = L_{g^{-1}*}X$, $X \in T_gG$. The decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ gives rise to another decomposition $\Omega = \Omega_{\mathfrak{t}} \oplus \Omega_{\mathfrak{m}}$. Write $\Omega_{\mathfrak{m}} = \Sigma\Omega^{2i-1} \otimes E_{2i-1} + \Omega^{2i} \otimes E_{2i}$. So $\Omega^{2i-1} \otimes E_{2i-1} + \Omega^{2i} \otimes E_{2i}$ is the V_i-component of $\Omega_{\mathfrak{m}}$.

As before let $J = \bigoplus \Sigma\varepsilon_i J_i$ denote an invariant almost complex structure on G/T. Complexify $\mathfrak{m}$, $\mathfrak{m}_C = \mathfrak{m} \otimes C = \bigoplus \Sigma V_i \otimes C$. There is the type decomposition $V_i \otimes C = V_i^{(1,0)} \oplus V_i^{(0,1)}$. $V_i^{(1,0)}$ (respectively $V_i^{(0,1)}$) is the $\sqrt{-1}$ (respectively $-\sqrt{-1}$) eigenspace of $J \otimes C$ restricted to $V_i \otimes C$.

PROPOSITION. *The type $(1,0)$ forms of G/T relative to J are given by C-linear combinations of the pullbacks of $\omega^i = \Omega^{2i-1} + \sqrt{-1}\varepsilon_i\Omega^{2i}$.*

PROOF. A simple computation shows that $E_{2j-1} + \sqrt{-1}\varepsilon_j E_{2j} \in V_j^{(0,1)}$, $1 \le j \le n$. At the identity $\omega^i(E_{2j-1} + \sqrt{-1}\varepsilon_j E_{2j}) = 0$, $1 \le i, j \le n$. The invariance of J and the left-invariance of ω do the rest. Q.E.D.

The following formulation of the integrability of J may be taken as a definition (e.g., [**Ch3**, p. 15]).

J is integrable if $d\omega^i = 0 \pmod{\omega}$ on G/T.

We now introduce an important notion, that of the horizontal distribution of G/T. There are $r = \operatorname{rank} G = \dim T$ many simple roots of G. Let $\Delta_s = \{\varepsilon_s\theta_s\} \subset \Delta_+$ denote the set of simple roots. Define $\mathfrak{s} = \bigoplus \Sigma V_s \subset \mathfrak{m}$. Restricting $g_* \circ \pi_{*e}\colon \mathfrak{m} \to T_TG/T \to T_{gT}G/T$ to $\mathfrak{s}$ we obtain a $2r$-dimensional distribution, denoted

by $\mathfrak{s}(G/T)$, of G/T: to see that this distribution is well-defined just note that $\mathrm{Ad}_t\mathfrak{s} = \mathfrak{s}$ for every $t \in T$.

DEFINITION. $\mathfrak{s}(G/T)$ is called the horizontal distribution of G/T relative to J.

It is not hard to see that $\mathfrak{s}(G/T)$ actually defines a holomorphic distribution in $T^{(1,0)}G/T$.

Another way of defining the above horizontal distribution is the exterior system on G/T given by

$$\Sigma = \{\omega^a = 0, \text{ where } \varepsilon_a\theta_a \text{ is not a simple root}\}.$$

For a different description of $\mathfrak{s}(G/T)$, using the complexification of G, see [**Br**, §4].

2. The Gauss-Frenet map. Let $G = \mathrm{Sp}(n)$ and also let $T = U(1)^n$. (We think of the quaternions H as $\{z_1 + z_2 j\colon z_i \in C\}$. This induces the inclusion $U(1)^n \subset \mathrm{Sp}(n)$.) Recall that $G = \{X \in \mathrm{GL}(n;H)\colon {}^t\overline{X}X = I\}$ where $\overline{\ }$ denotes the quaternionic conjugation. $M = G/T$ is the $\mathrm{Sp}(n)$-flag manifold of real dimension $2n^2$.

There is the fibration $G/T \to \mathrm{Sp}(n)/(\mathrm{Sp}(1))^n = F_{1,2,\dots,n}(H^n)$ with the standard fibre $(\mathrm{Sp}(1)/U(1))^n = CP^1 \times \cdots \times CP^1$. $F_{1,2,\dots,n}(H^n)$ is the full quaternionic flag manifold in H^n.

We have the usual decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. $\mathfrak{t}$, the Lie algebra of T, consists of purely imaginary (real multiples of $\sqrt{-1}$) $n \times n$ diagonal matrices. Let $E_{\alpha\beta} =$ the $n \times n$ matrix with $+1$ at (α,β)-entry, -1 at (β,α)-entry and zeros elsewhere. Here $1 \le \alpha < \beta \le n$. Also let $F_{\alpha\beta} =$ the $n \times n$ matrix with $+1$ at (α,β) and (β,α)-entries and zeros elsewhere. Here $1 \le \alpha \le \beta \le n$.

Put $V_{\alpha\beta} = RE_{\alpha\beta} \oplus iRF_{\alpha\beta}$, $1 \le \alpha < \beta \le n$. Also put $V'_{\alpha\beta} = jRF_{\alpha\beta} \oplus kRF_{\alpha\beta}$, $1 \le \alpha \le \beta \le n$. Then $\mathfrak{m} = \bigoplus \Sigma V_{\alpha\beta}$ $(1 \le \alpha < \beta \le n) \oplus \Sigma V'_{\alpha\beta}$ $(1 \le \alpha \le \beta \le n)$. Let $S(n;C)j$ denote the set of all complex symmetric $n \times n$ matrices multiplied on the right by j (j is a unit quaternion). Then we can write $\mathfrak{m} = (\mathfrak{u}(n)\backslash\mathfrak{t}) \oplus S(n;C)j$.

Let $t = (e^{it_\alpha}) \in T = U(1)^n$. The adjoint representation of T on $\mathfrak{m}$ is given by

$$\begin{aligned}\mathrm{Ad}_t\colon v_{\alpha\beta} &= xE_{\alpha\beta} + iyF_{\alpha\beta}\ (\in V_{\alpha\beta}) \mapsto e^{i(t_\alpha - t_\beta)}v_{\alpha\beta},\\ v'_{\alpha\beta} &= xF_{\alpha\beta}j + iyF_{\alpha\beta}j\ (\in V'_{\alpha\beta}) \mapsto e^{i(t_\alpha + t_\beta)}v'_{\alpha\beta}.\end{aligned}$$

Now $\mathfrak{t} \cong R^n$ via $\mathrm{diag}(ix_1,\dots,ix_n) \mapsto (x_1,\dots,x_n)$. Then the roots ($\subset \mathfrak{t}^*$) are $\{\pm 2x_\alpha, \pm(x_\alpha + x_\beta), \pm(x_\alpha - x_\beta), 1 \le \alpha < \beta \le n\}$. For positive roots we take $\Delta_+ = \{2x_\alpha, x_\alpha + x_\beta, x_\alpha - x_\beta, 1 \le \alpha < \beta \le n\}$. Then the simple roots are $\Delta_s = \{x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n, 2x_n\}$.

$\Omega = (\Omega^\alpha_\beta)$ denotes the $\mathfrak{sp}(n)$-valued Maurer-Cartan form of G. We note that $\mathfrak{sp}(n)$ consists of $n \times n$ H-valued matrices with ${}^t\overline{X} = -X$ and hence $\Omega^\alpha_\beta = -\overline{\Omega}^\beta_\alpha$.

Let $\Gamma^\alpha_\beta, \Sigma^\alpha_\beta$ be the complex-valued 1-forms with

$$\Omega^\alpha_\beta = \Gamma^\alpha_\beta + \Sigma^{\alpha_j}_\beta.$$

Then $\Omega^\alpha_\beta = -\overline{\Omega}^\beta_\alpha$ is equivalent to $\Gamma^\alpha_\beta = -\overline{\Gamma}^\beta_\alpha$, $\Sigma^\alpha_\beta = \Sigma^\beta_\alpha$. That is to say, Γ is $\mathfrak{u}(n)$-valued and Σ is $S(n;C)$-valued.

The Maurer-Cartan structure equations $d\Omega = -\Omega \wedge \Omega$ become

$$d\Gamma^\alpha_\beta = -\Gamma^\alpha_\gamma \wedge \Gamma^\gamma_\beta + \Sigma^\alpha_\gamma \wedge \overline{\Sigma}^\gamma_\beta, \qquad d\Sigma^\alpha_\beta = -\Gamma^\alpha_\gamma \wedge \Sigma^\gamma_\beta - \Sigma^\alpha_\gamma \wedge \overline{\Gamma}^\gamma_\beta.$$

Let J be the invariant (integrable) complex structure on G/T arising from the choice Δ_+, of positive roots. The corresponding "$(1,0)$-component" forms of $\Omega_{\mathfrak{m}}$ are $\theta^{\alpha\beta} = \Gamma^\alpha_\beta$, $1 \le \alpha < \beta \le n$, and $\theta'^{\alpha\beta} = \Sigma^\alpha_\beta$, $1 \le \alpha \le \beta \le n$.

For an invariant hermitian metric on G/T we take $-1/(2n+2)$ times the Cartan-Killing form restricted to $\mathfrak{m}$: $ds^2 = -\operatorname{tr}(\Omega_{\mathfrak{m}} \cdot \Omega_{\mathfrak{m}}) = \Sigma\Gamma^\alpha_\beta\overline{\Gamma}^\alpha_\beta$ $(1 \le \alpha < \beta \le n) + \Sigma\Sigma^\alpha_\beta\overline{\Sigma}^\alpha_\beta$ $(1 \le \alpha \le \beta \le n)$.

Let S be a connected Riemann surface. We consider a holomorphic immersion $f\colon S \to \mathrm{Sp}(n)/T$. If e is a local moving frame along f (i.e., a local lifting of f into $\mathrm{Sp}(n)$) then we define $Z^\alpha_\beta, Z'^\alpha_\beta$, local complex-valued functions on S by

$$\begin{aligned} e^*\Gamma^\alpha_\beta &= Z^\alpha_\beta\phi, \qquad 1 \le \alpha < \beta \le n, \\ e^*\Sigma^\alpha_\beta &= Z'^\alpha_\beta\phi, \qquad 1 \le \alpha \le \beta \le n, \end{aligned} \tag{1}$$

where ϕ is a local type $(1,0)$ coframe on S. (It is easily seen that the holomorphy of f allows us to do this.)

If $\tilde{e} = e \cdot t$ ($t = (e^{it_\alpha})$ is a local T-valued function on S) is another moving frame along f (all moving frames along f arise in this manner since $f^{-1}G$ is a T-principal bundle over S) then define tilded quantities by $\tilde{e}^*\Gamma^\alpha_\beta = \tilde{Z}^\alpha_\beta\phi$, $\tilde{e}^*\Sigma^\alpha_\beta = \tilde{Z}'^\alpha_\beta\phi$.

The adjoint representation of $T = U(1)^n$ quickly gives the transformation rules

$$\begin{aligned} \tilde{Z}^\alpha_\beta &= e^{i(t_\beta - t_\alpha)}Z^\alpha_\beta, \qquad 1 \le \alpha < \beta \le n, \\ \tilde{Z}'^\alpha_\beta &= e^{-i(t_\alpha + t_\beta)}Z'^\alpha_\beta, \qquad 1 \le \alpha \le \beta \le n. \end{aligned} \tag{2}$$

It follows that $r^\alpha_\beta = Z^\alpha_\beta\overline{Z}^\alpha_\beta$ and $r'^\alpha_\beta = Z'^\alpha_\beta\overline{Z}'^\alpha_\beta$ are globally defined invariants on S.

DEFINITION. A point $x \in S$ is called a regular point of f if none of the invariants $(r^\alpha_\beta), (r'^\alpha_\beta)$ vanishes at x. Otherwise x is called a singular point (of f).

NOTATION. $e^*\Gamma = \gamma$, $e^*\Sigma = \sigma$, etc.

Assume now that f is horizontal, i.e., f_* is tangent to the horizontal distribution. Recalling the exterior system at the end of §1 and consulting the simple roots Δ_s above we see that

$$\begin{aligned} \sigma^A_B &= 0, \qquad 1 \le A, B \le n, \qquad (A,B) \ne (n,n), \\ \gamma^C_D &= 0, \qquad 1 \le C < D \le n, \qquad D \ne C+1, \quad \text{on } S. \end{aligned} \tag{3}$$

Putting (1) and (3) together we get

$$\begin{aligned} \gamma^i_{i+1} &= a_i\phi, \ 1 \le i \le n-1, \qquad (a_i) \text{ local complex-valued functions on } S, \\ \sigma^n_n &= a_n\phi, \qquad a_n \text{ a local complex-valued function on } S, \\ \sigma^A_B &= 0, \quad \gamma^C_D = 0, \qquad A, B, C, D \text{ as in the above.} \end{aligned} \tag{4}$$

DEFINITION. Let U be a domain in S. A complex-valued smooth function $F\colon U \to S$ is said to be of analytic type if for each point $x \in U$, if z is a local holomorphic coordinate centered at x, then

$$F = z^b\tilde{F}, \tag{5}$$

where b is a positive integer and $\tilde{F}$ is a smooth complex-valued function with $\tilde{F}(x) \ne 0$.

It is known that functions of analytic type are exactly the solutions of exterior equation

$$(dF + \psi F) \wedge \phi = 0, \tag{6}$$

where ψ is a complex-valued 1-form on U and ϕ is a nowhere zero type $(1,0)$ form on U. So if F is a function of analytic type on U, then either F is identically zero on U or its zeros are isolated and of finite multiplicity (the integer b in (5) is the multiplicity at x).

Assigning S the metric induced by f we assume that f is an isometric immersion. Without loss of generality we take ϕ to be unitary. It follows that Σa_i^2 $(1 \leq i \leq n) \equiv 1$.

PROPOSITION. *$a_i \colon U$ (= the domain of e) $\to C$ is of analytic type for $i = 1$ to n.*

PROOF. Exterior differentiate both sides of the equations in the first two lines of (4) (using the Maurer-Cartan structure equations) and obtain

$$\begin{aligned} &[da_1 + ia_1\omega + a_1(\gamma_1^1 - \gamma_2^2)] \wedge \phi = 0, \\ &\qquad\vdots \\ &[da_{n-1} + ia_{n-1}\omega + a_{n-1}(\gamma_{n-1}^{n-1} - \gamma_n^n)] \wedge \phi = 0, \\ &[da_n + ia_n\omega + 2a_n\gamma_n^n] \wedge \phi = 0, \end{aligned} \tag{7}$$

where ω is the Levi-Civita connection form relative to ϕ. We see that $d\phi = i\omega \wedge \phi$ and the result follows from (6). Q.E.D.

Observe from (2) that near a regular point of f we can choose a moving frame e such that relative to e, $a_i > 0$ for every i. Hereafter we choose such a moving frame e along f.

Remember that (a_i^2) are all globally defined smooth functions on S and that Σa_i^2 $(1 \leq i \leq n) = 1$. Moreover, the above proposition tells us that each a_i^2 has only isolated zeros. It follows that the positive square root of each a_i^2 $(= a_i)$ is a function on S smooth away from the zeros and continuous at the zeros.

DEFINITION. Let $f \colon S \to \mathrm{Sp}(n)/T$ be a holomorphic horizontal isometric immersion from a connected Riemannian surface S. Then we define its Gauss-Frenet map $\phi_f \colon S \to S^{n-1} \subset R^n$ by $\phi_f = (a_i)$.

Note that the Gauss-Frenet map is continuous everywhere and is smooth away from the singular points of f.

CONVENTION. f as in the above definition will be called simply a horizontal holomorphic curve hereafter.

3. Frenet equations. For the rest of the paper we maintain the notation introduced in §2.

We first take care of the degenerate case thus simplifying the subsequent exposition.

DEFINITION. A horizontal holomorphic curve $f \colon S \to \mathrm{Sp}(n)/T$ is said to be degenerate if $a_i(f) \equiv 0$ for some i.

THEOREM. *Let $f\colon S \to \mathrm{Sp}(n)/T$ be a degenerate horizontal holomorphic curve. Then* (i) *if $a_n \equiv 0$ then $f(S)$ is congruent to an open submanifold of $U(n)/T =$ the* SU(n)*-flag manifold,* (ii) *if $a_i \equiv 0$ ($i \neq n$) then $f(S)$ is congruent to an open submanifold of* Sp(k)$/T'$, *where T' is a maximal torus of* Sp(k), $k < n$.

PROOF. This is an elementary application of the Frobenius theorem on completely integrable systems. We take $n = 3$, $a_1 \equiv 0$ and show that $f(S)$ (up to a congruence) $\subset \mathrm{Sp}(2)/T'$. The rest of the proof, being totally analogous, is omitted. Consulting the equations in (4) we see that $f(S)$ is congruent (i.e., differs by a fixed transformation in Sp(n)) to an integral manifold of the exterior system on G given by $\{\Gamma_1^1 = \Gamma_3^1 = 0,\ \Sigma_1^1 = \Sigma_2^1 = \Sigma_3^1 = \Sigma_2^2 = \Sigma_3^2 = 0\}$. This system is completely integrable and its analytic subgroup is easily seen to be a subgroup of $H = U(1) \times \mathrm{Sp}(2) \subset \mathrm{Sp}(3)$. Now a standard argument shows that $f(S)$ is congruent to an open submanifold of $H/T = \{e\} \times \mathrm{Sp}(2)/\{e\} \times U(1)^2 = \mathrm{Sp}(2)/T'$. Q.E.D.

From now on we exclude the degenerate curves from our consideration.

Let $f\colon S \to \mathrm{Sp}(n)$ be a (nondegenerate) horizontal holomorphic curve. There is the Gauss-Frenet map of f given by $\phi_f = (a_i)\colon S \to S^{n-1}$. A singular point x of f is where $a_i(x) = 0$ for some i. These points are isolated since (a_i^2) are all of analytic type. We recall the equations in (4).

As we saw the exterior differentation of (4) led to (7). Rewrite the equations in (7) as follows:

$$(8)\qquad \left[d\ \log a_j + i(\omega + i(\gamma_{j+1}^{j+1} - \gamma_j^j))\right] \wedge \phi = 0, \quad \text{where } 1 \le j \le n,\ \gamma_{n+1}^{n+1} = -\gamma_n^n.$$

Noting that $\omega + i(\gamma_{j+1}^{j+1} - \gamma_j^j)$, $\omega - 2i\gamma_n^n$ are all real we get

$$(9)\quad *d\ \log a_j = \omega + i(\gamma_{j+1}^{j+1} - \gamma_j^j), \quad \text{where } * \text{ is the Hodge star operator of } S, ds^2.$$

Let K denote the Gaussian curvature of (S, ds^2) so that $d\omega = (i/2)K\phi \wedge \overline{\phi}$. Also let Δ denote the Laplace-Beltrami operator of (S, ds^2) so that $d * d \log a = (i/2)\Delta \log a\phi \wedge \overline{\phi}$.

Using the Maurer-Cartan structure equations of Sp(n) coupled with (4) we obtain $d\gamma_1^1 = a_1^2\phi \wedge \overline{\phi}$, $d\gamma_2^2 = (a_2^2 - a_1^2)\phi \wedge \overline{\phi}, \ldots, d\gamma_n^n = (a_n^2 - a_{n-1}^2)\phi \wedge \overline{\phi}$. Exterior differentiations of both sides of the equations in (9) now give

$$(10)\qquad \tfrac{1}{2}(-K + \Delta\ \log a_j) = a_{j+1}^2 - 2a_j^2 + a_{j-1}^2, \quad \text{where } a_0 = 0,\ a_{n+1} = a_{n-1}.$$

The rank of the coefficient matrix relative to $(a_{n-1}^2, \ldots, a_1^2$, constant) of the right-hand sides of (10) is n. It follows that there exists exactly one affine relation among $\Delta \log a_1, \ldots, \Delta \log a_n$, and K. We write down this relation for $n = 2, 3$. (In general this relation is obtained by inverting the above mentioned coefficient matrix.)

$$(11\text{a})\qquad n = 2\colon\ \Delta\ \log a_1^4 a_2^3 = 7K - 4,$$

$$(11\text{b})\qquad n = 3\colon\ \Delta\ \log a_1^3 a_2^5 a_3^3 = 11K - 2.$$

COROLLARY. *Let $f\colon S \to \mathrm{Sp}(n)/T$ be a horizontal holomorphic curve from a compact surface S. Then* (i) *for $n = 2$, $K \ge 4/7$ implies that $K \equiv 4/7$, and* (ii) *for $n = 3$, $K \ge 2/11$ implies that $K \equiv 2/11$.*

PROOF. We will prove (i), (ii) being similar. The hypothesis implies that $\log a_1^4 a_2^3$ is a subharmonic function on S with singularities at the singular points

of f where it goes to $-\infty$. Since S is assumed to be compact $\log a_1^4 a_2^3$ must attain a maximum in S. The maximum principle for subharmonic functions does the rest. Q.E.D.

4. Integral formulae. Let S be a compact surface of genus g with Euler-Poincaré characteristic $\chi_S = 2 - 2g$. Also let $f\colon S \to \mathrm{Sp}(n)/T$ be a nondegenerate holomorphic horizontal isometric immersion where S is assigned a Riemannian metric in its conformal class. Then in $S\backslash\{\text{the singular points of } f\}$ the equations in (10) hold. Note that $\{\text{the singular points of } f\} = \bigcup a_j^{-1}\{0\}$ $(1 \le j \le n)$ and that this set is finite.

We have $\mathrm{area}(S) = (i/2)\int_S \phi \wedge \overline{\phi}$. Note that though ϕ is defined only locally $\phi \wedge \overline{\phi}$ is a global two-form and $(i/2)\phi \wedge \overline{\phi}$ is the area form of S.

The Gauss-Bonnet-Chern theorem states that $\chi_S = (i/4\pi)\int_S K\phi \wedge \overline{\phi}$.

As an elementary application of the argument principle we get

$$\frac{-i}{4\pi}\int_S \Delta \log a_j \phi \wedge \overline{\phi} = \#(a_j) \qquad (j = 1, \ldots, n),$$

where $\#(a_j)$ is the number of zeros of a_j each counted with multiplicity. This number is finite in view of the proposition in §2.

Integrations of both sides of (10), (11a), (11b) over S now give

THEOREM. *Let f be as in the above. Then*

$$\#(a_j) + \chi_S = \frac{1}{2\pi i}\int_S (a_{j+1}^2 - 2a_j^2 + a_{j-1}^2)\phi \wedge \overline{\phi}, \tag{12}$$

$$\textit{where } 1 \le j \le n,\ a_0 = 0,\ a_{n+1} = a_{n-1},$$

$$\textit{for } n = 2, \qquad 4\#(a_1) + 3\#(a_2) + 7\chi_S = \frac{2}{\pi}\mathrm{area}(S), \tag{12a}$$

$$\textit{for } n = 3, \qquad 3\#(a_1) + 5\#(a_2) + 3\#(a_3) + 11\chi_S = \frac{1}{\pi}\mathrm{area}(S). \tag{12b}$$

The above formulae may be considered as a sort of "Plücker relations".

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