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THE CARDINALITY OF REDUCED POWER SET ALGEBRAS

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ABSTRACT. We prove a general result on the cardinality of reduced powers of structures via filters that has several consequences including the following: if I is a uniform, countably complete ideal on the real line $\mathscr{R}$ and $\mathscr{B}$ is the Boolean algebra of subsets of $\mathscr{R}$ modulo I, then $|\mathscr{B}| > 2^{\aleph_0}$ and if $2^\nu \leq 2^{\aleph_0}$ for all $\nu < 2^{\aleph_0}$ then $|\mathscr{B}| = 2^{2^{\aleph_0}}$. This strengthens some results of Kunen and Pelc [**7**] and Prikry [**8**] obtained by Boolean ultrapower techniques. Our arguments are all combinatorial and some applications are included.

1. Introduction. Our notation and terminology are fairly standard. If A and B are sets then ${}^AB$ denotes the set of all functions mapping A into B and $\mathscr{P}(A)$ denotes the power set of A. Throughout, κ and λ will denote infinite cardinals. An ideal I on κ is a collection of subsets of κ that is closed under subset formation and finite unions. All the ideals we will consider are assumed to be proper ($I \neq \mathscr{P}(\kappa)$) but not necessarily κ-complete (i.e. closed under unions of size less than κ) or even uniform (i.e. containing all subsets of κ of size less than κ). If I is an ideal on κ then I^* denotes the filter on κ that is dual to I; that is, $X \in I^*$ iff $\kappa - X \in I$, and I^+ denotes $\mathscr{P}(\kappa) - I$.

If I is an ideal on κ and B is an arbitrary set then the reduced power of B (via I^*), denoted $\prod_\kappa B/I^*$, is the set of all equivalence classes of functions in ${}^\kappa B$ where we identify two such functions f and g if $\{\alpha < \kappa\colon f(\alpha) \neq g(\alpha)\} \in I$. The Boolean algebra $\mathscr{P}(\kappa)/I$ is the set of all equivalence classes of subsets of κ with the induced ordering from $\subseteq$, where we identify two such subsets X and Y if their symmetric difference $X \Delta Y$ is in I.

Since we will be comparing $\mathscr{P}(\kappa)/I$ and $\prod_\kappa \kappa/I^*$ it is worth pointing out the following (see [**3** or **1**, p. 338]). The function which associates to each $X \subseteq \kappa$ its characteristic function induces a natural bijection between $\mathscr{P}(\kappa)/I$ and $\prod_\kappa 2/I^*$ where, of course, $2 = \{0,1\}$. Hence $|\mathscr{P}(\kappa)/I| = |\prod_\kappa 2/I^*| \leq |\prod_\kappa \kappa/I^*|$. In general, the inequality cannot be replaced by equality since trivially $|\prod_\kappa \kappa/I^*| \geq \kappa$ (consider the constant functions from κ to κ) but $|\mathscr{P}(\kappa)/I| = 2$ if I is a prime (i.e. maximal) ideal on κ.

2. The main results. At the heart of our first theorem is the following trivial observation: there is a natural bijection between κ-sequences of subsets of λ and λ-sequences of subsets of κ. This correspondence is given by $(X_\alpha\colon \alpha < \kappa) \leftrightarrow (Y_\beta\colon \beta < \lambda)$ where $\alpha \in Y_\beta$ iff $\beta \in X_\alpha$. The proof of the following result consists of simply verifying that if I is a λ^+-complete ideal on κ then this correspondence induces a (well-defined) bijection between $\prod_\kappa \mathscr{P}(\lambda)/I^*$ and ${}^\lambda(\mathscr{P}(\kappa)/I)$.

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THEOREM 2.1. *If I is a λ^+-complete ideal on κ then*

$$\left|\prod_\kappa {}^\lambda 2/I^*\right| = \left|\prod_\kappa 2/I^*\right|^\lambda .$$

PROOF. Define $\phi\colon \prod_\kappa {}^\lambda 2/I^* \to {}^\lambda(\prod_\kappa 2/I^*)$ as follows: if $[f] \in \prod_\kappa {}^\lambda 2/I^*$, then $f\colon \kappa \to {}^\lambda 2$ and we set

$$\phi([f]) = \langle [f_\beta]\colon \beta < \lambda\rangle$$

where $f_\beta\colon \kappa \to 2$ is given by $f_\beta(\alpha) = f(\alpha)(\beta)$. Note that $[f_\beta] \in \prod_\kappa 2/I^*$.

To complete the proof of the theorem it suffices to show that ϕ is a well-defined bijection.

CLAIM 1. ϕ is well defined.

PROOF. Suppose that $f, g\colon \kappa \to {}^\lambda 2$ and $[f] = [g]$. Let $X = \{\alpha < \kappa\colon f(\alpha) = g(\alpha)\}$. Then $X \in I^*$. Fix $\beta < \lambda$ and let $Y_\beta = \{\alpha < \kappa\colon f_\beta(\alpha) = g_\beta(\alpha)\}$. We claim that $X \subseteq Y_\beta$. That is, if $\alpha \in X$ then $f(\alpha) = g(\alpha)$ so $f(\alpha)(\beta) = g(\alpha)(\beta)$. But then $f_\beta(\alpha) = g_\beta(\alpha)$ so $\alpha \in Y_\beta$ as desired. But now since $X \in I^*$ and $X \subseteq Y_\beta$ we have $Y_\beta \in I^*$ and so $[f_\beta] = [g_\beta]$. Since β was arbitrary this shows that $\langle [f_\beta]\colon \beta < \lambda\rangle = \langle [g_\beta]\colon \beta < \lambda\rangle$. Thus $\phi([f]) = \phi([g])$ and so ϕ is well defined as claimed.

CLAIM. ϕ is one-to-one.

PROOF. Suppose that f, $g\colon \kappa \to {}^\lambda 2$ and assume that $\phi([f]) = \phi([g])$. Then for every $\beta < \lambda$ we have $[f_\beta] = [g_\beta]$ and so $Y_\beta \in I^*$ where $Y_\beta = \{\alpha < \kappa\colon f_\beta(\alpha) = g_\beta(\alpha)\}$. Let $Y = \bigcap_{\beta<\lambda} Y_\beta$. Since I is λ^+-complete we have $Y \in I^*$. Now let $X = \{\alpha < \kappa\colon f(\alpha) = g(\alpha)\}$. We claim that $Y \subseteq X$. That is, if $\alpha \in Y$ then for every $\beta < \lambda$ we have $\alpha \in Y_\beta$, and so $f_\beta(\alpha) = g_\beta(\alpha)$. But then for every $\beta < \lambda$ we have $f(\alpha)(\beta) = g(\alpha)(\beta)$ and so the functions $f(\alpha)$ and $g(\alpha)$ are equal. Hence $\alpha \in X$. Now, since $Y \in I^*$ and $Y \subseteq X$ we have $X \in I^*$ and so $[f] = [g]$ as desired.

CLAIM 3. ϕ is onto.

PROOF. Suppose $\langle [f_\beta]\colon \beta < \lambda\rangle \in {}^\lambda \prod_\kappa 2/I^*$. Define $f\colon \kappa \to {}^\lambda 2$ by $f(\alpha)(\beta) = f_\beta(\alpha)$. Then $[f] \in \prod_\kappa {}^\lambda 2/I^*$ and clearly $\phi([f]) = \langle [f_\beta]\colon \beta < \lambda\rangle$. □

Specializing Theorem 2.1 to the case where $\kappa = 2^\lambda$ and making use of some known results yields the following.

THEOREM 2.2. *Suppose I is a λ^+-complete ideal on $\kappa = 2^\lambda$. Then*
(i) $|\mathscr{P}(\kappa)/I|^\lambda = |\prod_\kappa \kappa/I^*| \geq \kappa$, *and*
(ii) *if I is also uniform (i.e. $[\kappa]^{<\kappa} \subseteq I$), then $|\mathscr{P}(\kappa)/I| > \kappa$.*

PROOF. Part (i) follows immediately from Theorem 2.1 and our comments regarding $\mathscr{P}(\kappa)/I$ and $\prod_\kappa 2/I^*$. For part (ii) we need the following well-known result (see [**3**, p. 204]): for every infinite cardinal γ there exists a family $\mathscr{F} \subseteq {}^\gamma\gamma$ with $|\mathscr{F}| = \gamma^+$ and $|\{\alpha < \gamma\colon f(\alpha) = g(\alpha)\}| < \gamma$ whenever $f, g \in \mathscr{F}$ and $f \neq g$. An immediate consequence of this is the fact that if I is a uniform ideal on κ then $|\prod_\kappa \kappa/I^*| \geq \kappa^+$. To complete the proof of (ii), let $\mu = |\mathscr{P}(\kappa)/I|$ and suppose for contradiction that $\mu \leq \kappa$. Then

$$\kappa^+ \leq \left|\prod_\kappa \kappa/I^*\right| = |\mathscr{P}(\kappa)/I|^\lambda = \mu^\lambda \leq \kappa^\lambda = (2^\lambda)^\lambda = 2^\lambda = \kappa. \quad \square$$

Finally, if we specialize still further and again use some known results we obtain the following.

THEOREM 2.3. *Suppose I is an ω_1-complete ideal on the real line $\mathscr{R}$. Then*
(i) $|\mathscr{P}(\mathscr{R})/I| = |\prod_{\mathscr{R}} \mathscr{R}/I^*| = |\prod_{\mathscr{R}} \mathscr{R}/I^*|^{\omega}$,
(ii) *if I is uniform then $|\mathscr{P}(\mathscr{R})/I| > 2^{\aleph_0}$, and*
(iii) *if I is uniform and $2^{\nu} \leq 2^{\aleph_0}$ for all $\nu < 2^{\aleph_0}$ then $|\mathscr{P}(\mathscr{R})/I| = 2^{2^{\aleph_0}}$.*

PROOF. Part (i) follows immediately from Theorem 2.2(i) and the following result of Comfort-Hager [**2**] and Monk-Sparks [**6**]: if B is an infinite ω_1-complete Boolean algebra, then $|B|^{\omega} = |B|$. Since $\mathscr{P}(\mathscr{R})/I$ is ω_1-complete and infinite, this gives us that $|\mathscr{P}(\mathscr{R})/I| = |\mathscr{P}(\mathscr{R})/I|^{\omega}$.

Part (ii) is an immediate consequence of Theorem 2.2(ii).

Part (iii) requires the following well-known result of Tarski (see [**3**, p. 204]): if γ is an infinite cardinal and $2\mathfrak{I} = \gamma$ then there exists a family $\mathscr{F} \subseteq {}^{\gamma}\gamma$ with $|\mathscr{F}| = 2^{\gamma}$ and $|\{\alpha < \gamma \colon f(\alpha) = g(\alpha)\}| < \gamma$ whenever $f, g \in \mathscr{F}$ and $f \neq g$. (To see this, let $\langle X^{\alpha}_{\beta} \colon \beta < \gamma_{\alpha}\rangle$ be an enumeration of $\mathscr{P}(\alpha)$ in order-type $\gamma_{\alpha} \leq \gamma$ for each $\alpha < \gamma$ and for each $X \subseteq \gamma$ define $f_X \colon \gamma \to \gamma$ by $f_X(\alpha) = \beta$ iff $X \cap \alpha = X^{\alpha}_{\beta}$.) In the context of (iii), with $\gamma = 2^{\aleph_0}$ $(= |\mathscr{R}|)$, this gives us that $|\prod_{\mathscr{R}} \mathscr{R}/I^*| \geq 2^{2^{\aleph_0}}$, and so (iii) now follows from (i). □

The second equality in part (i) of Theorem 2.3 should be compared with Keisler's result [**5**] that $|\prod \mathscr{R}/D| = |\prod \mathscr{R}/D|^{\omega}$ for any countably incomplete ultrafilter D on ω. Part (ii) of Theorem 2.3 eliminates some saturation requirements from an old (unpublished) result of Kunen (recently rediscovered by Pelc [**7**]) and part (iii) does the same for a result of Prikry [**8**]. The proofs of Kunen, Pelc and Prikry all employed Boolean ultrapowers.

3. Some applications. We conclude with a few applications of the results in §2.

APPLICATION 1. We begin with an application that is due as much to Andrzej Pelc (see [**7**]) as to the present author. It answers the following question that we asked in [**9**, Problem R, p. 51]: Suppose we have countably many ω_1-complete uniform ideals on the real line $\mathscr{R}$. Can we then find a subset X of $\mathscr{R}$ such that neither X nor $\mathscr{R} - X$ belongs to any of the ideals in our collection? An affirmative answer to this can be derived as follows. By Theorem 5.4.2 of [**9**], it suffices to show that if I is an ω_1-complete uniform ideal on $\mathscr{R}$ then $\mathscr{P}(\mathscr{R})/I$ has no countable dense set (in the forcing theoretic sense). Assume we have such an I and $\{X_n \colon n \in \omega\} \subseteq I^+$. To each distinct $[Y] \in \mathscr{P}(\mathscr{R})/I$ we associate $\{n \in \omega \colon [X_n] \leq [Y]\}$. Since $|\mathscr{P}(\mathscr{R})/I| > 2^{\aleph_0}$ by Theorem 2.3(ii), we get $[Y_1], [Y_2] \in \mathscr{P}(\mathscr{R})/I$ with $[Y_1] = [Y_2]$ and $[X_n] \leq [Y_1]$ iff $[X_n] \leq [Y_2]$ for every $n \in \omega$. But this shows that $\{X_n \colon n \in \omega\}$ is not a dense set since either $Y_1 - Y_2 \in I^+$ or $Y_2 - Y_1 \in I^+$ and neither contains X_n (modulo I) for any $n \in \omega$.

APPLICATION 2. Theorem 2.3(ii) also yields yet another proof that not every set of reals is Lebesgue measurable. To see this, assume the contrary and let $I = \{X \subseteq \mathscr{R} \colon \mu(X) = 0\}$. Then I is clearly ω_1-complete and it is uniform since every set of reals of cardinality less than the continuum has outer measure zero. Theorem 2.3(ii) now implies that $|\mathscr{P}(\mathscr{R})/I| > 2^{\aleph_0}$, and this contradicts the well-known fact that every measurable set differs from a Borel set by a set of measure zero (since there are only $2^{\aleph_0}$ Borel sets).

APPLICATION 3. We showed above that a consequence of Theorem 2.3 is the fact that no ω_1-complete uniform ideal I on $\mathscr{R}$ can yield a Boolean algebra $\mathscr{P}(\mathscr{R})/I$

with a countable dense set. A natural question (again related to versions of Ulam's problem in [**9** and **10**]) is whether a dense set of size $\aleph_1$ is possible. Woodin [**11**] has shown that relative to large cardinals an affirmative answer is consistent. In his model CH holds. As a final application of Theorem 2.2 we consider the question of whether CH is necessary and prove the following: if $\mathscr{R}$ carries an ω_1-complete uniform ideal I so that $\mathscr{P}(\mathscr{R})/I$ has a dense set of size $\aleph_1$ then either $2^{\aleph_0} = \aleph_1$ or $2^{\aleph_0} > \aleph_{\omega_1}$. To see this we argue as follows. Suppose I is an ω_1-complete, uniform ideal on $\mathscr{R}$ and D is a dense set of size $\aleph_1$ for $\mathscr{P}(\mathscr{R})/I$. Define $\phi\colon \mathscr{P}(\mathscr{R})/I \to \mathscr{P}(D)$ by $\phi([A]) = \{d \in D\colon d \leq [A]\}$. It is easy to see that ϕ is one-to-one and so $|\mathscr{P}(\mathscr{R})/I| \leq 2^{\aleph_1}$. On the other hand, Theorem 2.2(ii) guarantees that $|\mathscr{P}(\mathscr{R})/I| > 2^{\aleph_0}$, and so $2^{\aleph_0} < 2^{\aleph_1}$. Now, if κ is the exact completeness of I it is easy to see that I projects to a κ-complete ideal on κ, and since I has a dense set of size of ω_1 it is clearly ω_2-saturated. Thus, if $2^{\aleph_0} < \aleph_{\omega_1}$ we must have $\kappa = \omega_1$ and so we arrive at a situation where we have an ω_2-saturated ω_1-complete ideal on ω_1 with $2^{\aleph_0} < 2^{\aleph_1}$. A well-known result of Jech-Prikry [**4**, p. 3] now yields $2^{\aleph_0} = \aleph_1$ as desired.

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