MORE NONDEVELOPABLE SPACES IN MOBI

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ABSTRACT. We construct two spaces with a point-countable base, the first is not weakly $\theta$-refinable, while the second is not a $\sigma\#$-space. Both spaces can be represented as open and compact images of metacompact developable Hausdorff spaces. This answers questions of H. R. Bennett and R. F. Gittings.

The first example of a nonweakly $\theta$-refinable space with a point-countable base was constructed, with the use of the continuum hypothesis, in [GG]. In the first section of this paper we apply an idea from [GG] in order to get a ZFC example of such a space. Our construction is simpler than the example of a metalindelöf space which is not weakly $\theta$-refinable from [Gr].

In the second section we show that the space constructed in the first section is an open and compact image of a metacompact developable Hausdorff space. Thus this space is an element of $\text{MOBI}_2$, where $\text{MOBI}_i$ is the minimal class of $T_i$-spaces containing all metric spaces and invariant under open and compact mappings.

In the third section we apply the techniques of the first two sections in order to construct a space in $\text{MOBI}_2$ which is not a $\sigma\#$-space.

1. A nonweakly $\theta$-refinable space with a point-countable base. It will be convenient to use the following definition [BL, Lemma 4]: a space $K$ is weakly $\theta$-refinable if every open cover of $K$ has a refinement $\mathcal{A} = \bigcup\{\mathcal{A}_n : n \geq 1\}$ such that each $\mathcal{A}_n$ is discrete in $\bigcup_{n=1}^{\infty} \mathcal{A}_n$.

Our construction will be based on an example of a space with a point-countable base which does not have a $\sigma$-point finite base [A]. The points of the space are functions from countable ordinals into $\omega$ and the base of neighborhoods of a function $s$ consists of sets of the form

$$B(s,m) = \{t \supseteq s : t = s \text{ or } t(\text{dom } s) \geq m\}.$$ 

Let $S$ be the space described above and put $B(s) = B(s,0)$ for $s \in S$. Clearly, every nonempty open subset of $S$ contains a set $B(s)$ for an $s \in S$ and any intersection of a decreasing sequence of sets of the form $B(s)$ is again a set of this form. In particular, $S$ is a Baire space.

Every open cover of the space $S$ has a disjoint open refinement [A, M]. We shall destroy the covering properties of $S$ with the help of the space $\omega_1$ of the countable ordinals (see [GG]).

Put

$$S^* = \{(\alpha, s) \in \omega_1 \times S : \alpha \leq \text{dom } s\}$$

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and consider $S^*$ with the topology inherited from $\omega_1 \times S$. We shall show that $S^*$ is not weakly $\theta$-refinable.

Let $U_\delta = \{ (\alpha, s) \in S^* : \alpha \leq \delta \}$ and $\mathcal{U} = \{ U_\delta : \delta \in \omega_1 \}$. Suppose that $S^*$ is weakly $\theta$-refinable, then there exists a refinement $\mathcal{A} = \bigcup \{ \mathcal{A}_n : n \geq 1 \}$ of $\mathcal{U}$ such that each $\mathcal{A}_n$ is discrete in $\bigcup \mathcal{A}_n$.

By induction on $\alpha \in \omega_1$ construct sequences $\langle s_\alpha \rangle_{\alpha \in \omega_1}$ of elements of $S$, $\langle n_\alpha \rangle_{\alpha \in \omega_1}$ of natural numbers and $\langle \gamma_\alpha \rangle_{\alpha \in \omega_1}$ of countable ordinals such that for each $\alpha > 0$,

(i) $s_\alpha \supset \bigcup \{ s_\beta : \beta < \alpha \}$ and $(\alpha, s_\alpha) \in S^*$,

(ii) $\big\{ \alpha \times B(s) \big\} \cap \bigcup \mathcal{A}_{n_\alpha} \neq \emptyset$ for any $s \supset s_\alpha$,

(iii) $\gamma_\alpha < \alpha$ and $(\gamma_\alpha, \alpha) \times B(s_\alpha)$ intersects at most one element of $\mathcal{A}_{n_\alpha}$.

Start by choosing $s_0 = \emptyset$, $n_0 = 0$ and $\gamma_0 = 0$. Suppose that $s_\beta$, $n_\beta$ and $\gamma_\beta$ have been defined for $\beta < \alpha$.

We shall define $s_\alpha$ by consecutively extending $\bigcup \{ s_\beta : \beta < \alpha \}$. First choose $s' \supset \bigcup \{ s_\beta : \beta < \alpha \}$ with $\text{dom } s' \geq \alpha$. Clearly, any extension $s_\alpha$ of $s'$ will satisfy (i). Since $\{ \alpha \} \times B(s')$ is covered by $\{ \bigcup \mathcal{A}_n : n \geq 1 \}$, one can use the Baire property of $S$ in order to find an $n_\alpha \geq 1$ and an $s'' \supset s'$ such that $\bigcup \mathcal{A}_{n_\alpha}$ is dense in $\{ \alpha \} \times B(s'')$.

Clearly, any extension $s_\alpha$ of $s''$ will satisfy (ii) with respect to this $n_\alpha$. In particular, there exists an $s''' \supset s''$ such that $(\alpha, s''') \in \bigcup \mathcal{A}_{n_\alpha}$ and, since $\mathcal{A}_{n_\alpha}$ is discrete in $\bigcup \mathcal{A}_{n_\alpha}$, $(\alpha, s''')$ has a neighborhood of the form $(\gamma_\alpha, \alpha) \times B(s''', m)$ intersecting exactly one element of $\mathcal{A}_{n_\alpha}$. Clearly, $s_\alpha = s''' \cup \{ (\text{dom } s'''', m) \}$ satisfies (iii) with respect to these $n_\alpha$ and $\gamma_\alpha$ and the inductive step is complete.

Observe that (ii) and (iii) imply that for each $\beta > 0$, there exists an $A_\beta \in \mathcal{A}_{n_\beta}$ which intersects $\{ \beta \} \times B(s)$ for all $s \supset s_\beta$.

In order to obtain a contradiction, use standard properties of $\omega_1$ to find an $n \geq 1$, $\gamma \in \omega_1$ and uncountable set $E \subseteq \omega_1$ such that $n_\alpha = n$ and $\gamma_\alpha = \gamma$ for $\alpha \in E$. Pick a $\beta \in E$ and a $\delta \in \omega_1$ such that $A_\beta \subseteq U_\delta$. If $\alpha \in E$ and $\alpha > \beta$, then $A_\alpha \neq A_\beta$ while $(\gamma, \delta) \times B(s_\alpha) \supset \{ \beta \} \times B(s_\beta)$, so $(\gamma, \alpha) \times B(s_\alpha)$ intersects two different elements $A_\alpha$ and $A_\beta$ of $\mathcal{A}_n$, which contradicts (iii).

It is easy to see that the space $S^*$ has a point-countable base. In fact, it is a first-countable space with a co-countable neighbournet.

2. Spaces with a co-countable neighbournet. Recall that a neighbournet for a space $Y$ is a relation $V \subseteq Y \times Y$ such that for each $y \in Y$, $y \in \text{Int } V(y)$, where $R(y) = \{ y' \in Y : (y, y') \in R \}$ for a relation $R \subseteq Y \times Y$ [J1]. We shall only consider neighbournets $V$ such that $V(y)$ is open for $y \in Y$. The neighbournet $V$ is called co-countable (co-finite) if $V^{-1}(y)$ is countable (finite) for $y \in Y$ [J2].

It is easy to check that first-countable $T_1$-spaces with a co-countable (co-finite) neighbournet have a point-countable base (are $\sigma$-discrete metacompact and developable [J2]).

Observe that the space $S^*$ has a co-countable neighbournet. Indeed, the collection $\{ B(\alpha, s) : (\alpha, s) \in S^* \}$, where $B(\alpha, s) = [0, \alpha] \times B(s)$, defines a co-countable neighbournet in $S^*$, for $(\beta, t) \in B(\alpha, s)$ implies that $s \subset t$ and $\alpha < \text{dom } s \leq \text{dom } t$.

In [J2, Theorems 3, 4] (first-countable) $T_1$-spaces with co-countable neighbournets have been characterized as images of "nice" spaces under open countable-to-one mappings. We shall modify the construction of open mappings from [J2] in order to get open and compact mappings.
LEMMA. If $Y$ is a $T_1$-space with a co-countable neighbournet $V$, then there exists a Hausdorff space $X$ with a co-finite neighbournet and an open and compact countable-to-one mapping $f$ from $X$ onto $Y$. Moreover, if $Y$ is first-countable, then so is $X$.

PROOF. As in [J2, Lemma 5], let $I(Y, V)$ be the set of all injections $\sigma: \{0, \ldots, n\} \to Y$ such that $e(\sigma)$ defined as $\sigma(n)$ is in the intersection of all $V(\sigma(k))$, $k = 0, \ldots, n$. The latter restriction assures that $\{\sigma: e(\sigma) = y\} \subset \{\sigma: \sigma(\text{dom } \sigma) \subset V^{-1}(y)\}$ for $y \in Y$, hence the natural projection $e$ of $I(Y, V)$ onto $Y$ is countable-to-one. The basic open sets containing $\sigma$ are of the form

$$B(\sigma, U) = \{\tau \supset \sigma: \tau(\text{dom } \tau \setminus \text{dom } \sigma) \subset U\},$$

where $U$ is an open subset of $Y$ containing $e(\sigma)$.

Observe that, if $\tau \in B(\sigma, U) \cap B(\sigma', U')$, then $\sigma, \sigma' \subset \tau$ and, assuming that $\sigma \subset \sigma'$, $\sigma' \in B(\sigma, U)$.

Clearly, the projection $e: I(Y, V) \to Y$ is continuous, open [J2] and, from the observation above, it follows that the fibers of $e$ are discrete. Thus $I(Y, V)$ is a $T_1$-space and, again, the observation above, implies that it is a collectionwise (in fact, monotonically) normal space. Moreover, the collection $\{B(\sigma, Y): \sigma \in I(Y, V)\}$ defines a co-finite neighbournet on $I(Y, V)$.

We shall modify $I(Y, V)$ by adding a limit point to each infinite fiber of $e$.

Put $X = I(Y, V) \cup \{y \in Y: e^{-1}(y) \text{ is infinite}\}$ and consider $X$ with a topology such that $I(Y, V)$ is open in $X$, while for $y \in X \setminus I(Y, V)$ basic neighbourhoods of $y$ are of the form

$$B(y, U, F) = \{y\} \cup \bigcup\{B(\sigma, U): \sigma \in e^{-1}(y) \setminus F\},$$

where $U$ is an open subset of $Y$ containing $y$ and $F$ is a finite subset of $e^{-1}(y)$.

It is easy to check that $X$ is a Hausdorff space and the collection $\{B(\sigma, Y): \sigma \in I(Y, V)\} \cup \{B(y, Y, \emptyset): y \in X \setminus I(Y, V)\}$ defines a co-finite neighbournet on $X$. Clearly, if $Y$ is a first-countable space, then $X$ is first-countable too.

Define $f: X \to Y$ as the combination of $e$ and the identity on $X \setminus I(Y, V)$. Clearly, $f$ is a continuous open and compact countable-to-one mapping.

If $Y$ in the above construction is the space $S^*$, then one gets

EXAMPLE. An open and compact mapping of a $\sigma$-discrete metacompact developable Hausdorff space onto the space $S^*$ which is not weakly $\theta$-refinable.

The example answers Question c from [B2] repeated in [G, Question 1]. Furthermore, if the space $Y$ in the lemma is first-countable, then $I(Y, V)$ is developable and collectionwise normal, hence, it is then a metrizable space (see [J2]). Thus $X$ is then a locally metrizable $\sigma$-discrete metacompact space and, therefore, $X$ can be represented as an open finite-to-one image of a $\sigma$-discrete metrizable space. In particular, $S^*$ is in the class MOBI$_2$, which gives an answer to Question 1 from [B1].

The above reasoning and the fact that open mappings with separable fibers preserve the property of having a co-countable neighbournet (see [J2, Proposition 2]) give

THEOREM. For a Hausdorff space $Y$ the following conditions are equivalent:

(i) $Y$ is a first-countable space with a co-countable neighbournet,
(ii) \( Y \) is an open and compact countable-to-one image of a \( \sigma \)-discrete metacompact developable Hausdorff space,

(iii) \( Y \) is in the minimal class of Hausdorff spaces containing all \( \sigma \)-discrete metric spaces and invariant under open and compact mappings.

In the next section we give a method of constructing spaces with co-countable neighbourhoods based on the construction of the space \( S \) and spaces \( I(Y,V) \) and we use this method in order to answer the remaining part of Question 1 from [G] concerning the invariantness of \( \sigma^\# \)-spaces.

3. \( \sigma^\# \)-spaces. According to the standard terminology, a collection \( \mathcal{A} \) of subsets of a space \( Y \) will be called \( T_1(T_0) \)-separating if for each two distinct points \( y \) and \( y' \) of \( Y \) there exists an \( A \in \mathcal{A} \) such that \( y \notin A \) and \( y' \notin A \) (or \( y \in A \) and \( y' \notin A \)). The collection \( \mathcal{A} \) is closure-preserving (\( \sigma \)-closure-preserving) if

\[
\bigcup \{A \cap A' : A \in \mathcal{A}'\} = \bigcup \{A : A \in \mathcal{A}'\}
\]

for any subcollection \( \mathcal{A}' \) of \( \mathcal{A} \) (if \( \mathcal{A} \) can be represented as a union of countably many closure-preserving subcollections). The space \( Y \) is called \( \sigma^\# \)-space if it has a \( \sigma \)-closure-preserving \( T_1 \)-separating collection of closed sets.

We shall answer the part of Question 1 in [G] concerning \( \sigma^\# \)-spaces by constructing a Hausdorff first-countable space \( T \) with a co-countable neighbourhood which is not a \( \sigma^\# \)-space. By virtue of the results of the previous section, the space \( T \) is then in MOBI2.

One can check that the space \( S^\ast \) constructed in the first section is a \( \sigma^\# \)-space. Moreover, any space with a co-countable neighbourhood has a closure-preserving cover by countable closed sets [J2, Proposition 1] and one can modify this cover so that it is, in addition, a \( T_0 \)-separating collection (see 4.2). The example that we are going to construct shows that this does not imply the existence of a \( \sigma \)-closure-preserving \( T_1 \)-separating closed collection.

**Example.** A first-countable Hausdorff space \( T \) with a co-countable neighbourhood which is not a \( \sigma^\# \)-space.

Let \( Z = C_1 \cup C_2 \) be the Alexandroff double circle. As in [E, 3.1.26], \( C_1 \) is the topological circle, \( C_2 \) is the set of isolated points of \( Z \), \( p \) is the projection of \( C_1 \) onto \( C_2 \) from the joint center of \( C_1 \) and \( C_2 \) and, for \( z \in C_1 \), the basic open sets containing \( z \) are of the form \( U_j(z) = V_j(z) \cup \{p(V_j(z) \setminus \{z\})\} \), where \( j \geq 1 \) and \( V_j(z) \) is an open arc of length \( 1/j \) in \( C_1 \) centered at \( z \).

We shall modify \( Z \) in order to obtain the space \( T \). The points of \( T \) will be one-to-one functions from nonlimit countable ordinals into \( Z \). For \( s \in T \) with \( \text{dom } s = \alpha + 1 \), by \( e(s) \) we denote \( s(\alpha) \). The basic open sets containing \( s \) are of the form

\[
B(s,U) = \{t \supseteq s : e(t) \in U\},
\]

where \( U \) is an open subset of \( Z \) containing \( e(s) \).

It is easy to see that \( T \) is a first-countable Hausdorff (but not regular) space and the collection \( \{B(s,Z) : s \in T\} \) defines a co-countable neighbourhood in \( T \).

Suppose that \( T \) is a \( \sigma^\# \)-space and let \( \mathcal{E} = \bigcup \{E_n : n \geq 1\} \) be a \( T_1 \)-separating collection of closed subsets of \( T \) such that each \( E_n \) is closure-preserving. Choose a one-to-one sequence \( \langle z_\beta \rangle_{\beta \in \omega_1} \) of elements of \( C_1 \) and denote \( s_\alpha = \langle z_\beta \rangle_{\beta \in \alpha} \in T \). For each \( \alpha \in \omega_1 \) find a decreasing sequence \( \{W_n(z_\alpha)\}_{n \geq 1} \) of open subsets
of $Z$ containing $z_\alpha = e(s_\alpha)$ such that each $W_n(z_\alpha) = U_j(z_\alpha)$ for a $j \geq n$ and $B(s_\alpha, W_n(z_\alpha)) \cap \{E \in \mathcal{Z}_n : s_\alpha \notin E\} = \emptyset$.

From the fact that $C_1$ is hereditarily separable, it follows that there exists an $\alpha \in \omega_1$ such that for each $\beta > \alpha$ and $n \geq 1$

$$C_1 \cap \bigcup\{W_n(z_\gamma) : \gamma \geq \alpha\} = C_1 \cap \bigcup\{W_n(z_\gamma) : \gamma > \beta\}.$$ 

In particular, for each $n \geq 1$, there exists a $\beta_n > \alpha$ such that $z_\alpha \in W_n(z_{\beta_n})$ and since each $W_n(z_\beta)$ is of the form $U_j(z_\beta)$ for $j > n$, we obtain that $p(z_\alpha) \in W_n(z_{\beta_n})$ and $(z_{\beta_n})_{n \geq 1}$ converges to $z_\alpha$ in $C_1$.

Let $\gamma$ be a countable ordinal greater than $\sup\{\beta_n : n \geq 1\}$ and define an $s \in T$ with dom $s = \gamma + 1$ by putting $s(\beta) = z_\beta$ for $\beta < \gamma$ and $s(\gamma) = p(z_\alpha) \in C_2$. If $n \geq 1$ and $E \in \mathcal{Z}_n$ are such that $s \in E$ and $s_\alpha \notin E$, then $B(s_\alpha, W_n(z_\alpha)) \cap E = \emptyset$. Find an $m \geq n$ for which $z_{\beta_m} \in W_n(z_\alpha)$. Clearly, $s_{\beta_m} \in B(s_\alpha, W_n(z_\alpha))$ and, consequently, $s_{\beta_m} \notin E$, which implies $B(s_{\beta_m}, W_n(z_{\beta_m})) \cap E = \emptyset$. However, $e(s) = p(z_\alpha) \in W_m(z_{\beta_m}) \subset W_n(z_{\beta_m})$, hence $s \in B(s_{\beta_m}, W_n(z_{\beta_m}))$ and this contradicts $s \notin E$.

4. Final remarks.

4.1. The space $S$ from the first section is an easy example of a non-quasi-developable space in MOBI$_2$. In fact, the space $S$ is not even a primitive $\sigma^\#$.space in the sense of [Ch2], which implies that $S$ is neither a $\sigma$-space nor a space with a primitive base. An easy way to see that $S$ is not a primitive $\sigma^\#$.space is to use a game-theoretic characterization of this property given, implicitly, in [M, Theorem 7.3].

Consider the following two-person game in a space $Y$: players I and II alternately choose nonempty subsets $C_1 \supset D_1 \supset C_2 \supset D_2 \supset \cdots$ of $Y$ such that each $D_n$ is relatively open in $C_n$. Player II (choosing $D_n$’s) wins if $\bigcap\{C_n : n \geq 1\}$ contains at most one point. The space $Y$ is a primitive $\sigma^\#$.space iff player II has a (stationary) winning strategy in this game. In $S$ player I has a stationary winning strategy, namely, I wins by choosing at each stage of the game a set of the form $B(s)$ contained in the set recently chosen by II.

Observe that $S$ is a $\sigma^\#$.space which is not a primitive $\sigma^\#$.space. This improves Example 1 in [Ch2] and shows that the term “primitive $\sigma^\#$.space” suggested in [Ch2] should be changed.

4.2. The following slight strengthening of Proposition 1 in [J2] (see the proof of (c) $\Rightarrow$ (a) in [F, Theorem 2.2]) clarifies the structure of $T_1$-spaces with a co-countable neighbournet:

**Proposition.** A $T_1$-space $Y$ has a co-countable neighbournet iff there exists a partial order $\preceq$ on $Y$ such that $\{y' : y' \preceq y\}$ is countable and $\{y' : y' \succeq y\}$ is open for $y \in Y$.

Observe that in the construction of $X$ in §2, one can consider only sequences $\sigma$ which are increasing with respect to the above partial order. The examples of spaces in MOBI$_3$ in [Ch1] were obtained in this way from spaces $Y$ with the partial order having two levels.

4.3. Theorem 2.2 in [F] gives several conditions characterizing spaces with a co-countable neighbournet in the class of $T_1$-spaces with a point-countable base. We give another such condition, which shows that the reflection principles $S_1$ and $S_2$ from [F] are equivalent (see Conjecture 1 in [F, 6]).
PROPOSITION. Let $Y$ be a $T_0$-space with a point-countable base $\mathcal{U}$. For each $y \in Y$ fix an enumeration $e(y) = (U_i(y))_{i \geq 1} \in \mathcal{U}^\omega$ (not necessarily one-to-one) of all the members of $\mathcal{U}$ containing $y$. Consider $\mathcal{U}^\omega$ with the Baire metric and let $M = e(Y) \subset \mathcal{U}^\omega$. Then $Y$ has a co-countable neighbournet iff $M$ is $\sigma$-discrete.

PROOF. Suppose that $Y$ has a co-countable neighbournet. Observe that $e$ is one-to-one and $e^{-1} : M \to Y$ is continuous. Clearly, this gives a co-countable neighbournet on $M$ and, since $M$ is a metric space, it is $\sigma$-discrete (see [F or J1, 4.8]).

In order to prove the other implication, assume that $M' = e(Y') \subset M$ is $1/j$-discrete and define $V(y') = \bigcap\{U_i(y') : i \leq j\}$ for $y' \in Y'$. If $y \in Y$ and $y \in V(y')$ for a $y' \in Y'$, then $U_i(y')$ is a term of $e(y)$ for $i \leq j$ and, consequently, the number of such $y' \in Y'$ is countable. Since $M$ is a countable union of such $M'$, this shows that $Y$ has a co-countable neighbournet.

4.4. The construction from [B3] showing that weak $\theta$-refinability is not invariant under open and compact mappings can be generalized as follows.

PROPOSITION. If a Hausdorff space $Y$ can be covered by an open collection $\mathcal{G}$ such that each element of $\mathcal{G}$ has a co-finite (co-countable) neighbournet, then $Y$ is an open and compact image of a Hausdorff space with a co-finite neighbournet.

PROOF. Suppose first that each $G \in \mathcal{G}$ has a co-finite neighbournet $V_G$. Let $e$ be the canonical mapping of the topological sum $X'$ of the elements of $\mathcal{G}$ onto $Y$. Put $X = X' \cup \{y \in Y : e^{-1}(y) \text{ is infinite}\}$ and consider $X$ with the topology such that $X'$ is open in $X$ and the basic open sets containing $y \in X \setminus X'$ are of the form

$$B(y, U, \mathcal{G}) = \{y\} \cup \{G \cap V_G(y) \cap U : y \in G \in \mathcal{G} \setminus \mathcal{G}\},$$

where $U$ is an open subset of $Y$ containing $y$ and $\mathcal{G}$ is a finite subcollection of $\mathcal{G}$.

It is easy to see that $X$ is a Hausdorff space with a co-finite neighbournet and the natural extension $f$ of $e$ is an open and compact mapping of $X$ onto $Y$.

If $W_G$ is a co-countable neighbournet in $G \in \mathcal{G}$, then $G$ in the above construction should be replaced with a space $X(G)$ constructed from $I(G, W_G)$ as in §2.

4.5. The nondevelopable Čech complete space with a point-countable base from [D] can be represented as an open and compact countable-to-one image of a meta-compact Moore space and, therefore, is in MOBI$_3$.

REFERENCES

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MORE NONDEVELOPABLE SPACES IN MOBI


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