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ON THE RELATIONSHIP OF AP, RS AND CEP IN CONGRUENCE MODULAR VARIETIES. II

CLIFFORD BERGMAN AND RALPH MCKENZIE

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ABSTRACT. Let V be a congruence distributive variety, or a congruence modular variety whose free algebra on 2 generators is finite. If V is residually small and has the amalgamation property, then it has the congruence extension property. Several applications are presented.

In two previous papers [**1** and **2**], we considered the following question: if $\mathscr{V}$ is a residually small variety with the amalgamation property, must $\mathscr{V}$ have the congruence extension property? Our work established the following implications for a congruence modular variety $\mathscr{V}$:

(1) If $\mathscr{V}$ is 2-finite and has C2, then $\mathrm{AP} + \mathrm{RS} \Rightarrow \mathrm{R}$.
(2) If $\mathscr{V}$ is 4-finite with C2 and R, then $\mathrm{AP} + \mathrm{RS} \Rightarrow \mathrm{CEP}$.

(The terminology will be explained below.)

In this paper we supplement and extend these results. Assuming still that $\mathscr{V}$ is congruence modular, we have:

(3) $\mathrm{AP} + \mathrm{RS} \Rightarrow \mathrm{C2}$.
(4) If $\mathscr{V}$ has R, then $\mathrm{AP} + \mathrm{RS} \Rightarrow \mathrm{CEP}$.

Combining these implications, we have that every congruence modular, 2-finite variety satisfies $\mathrm{AP} + \mathrm{RS} \Rightarrow \mathrm{CEP}$. Furthermore, every congruence distributive variety (no finiteness assumption) satisfies $\mathrm{AP} + \mathrm{RS} \Rightarrow \mathrm{CEP}$.

Our universal algebraic notation and terminology are standard. Good references are [**4** and **9**].

Let $\mathscr{V}$ be a variety of algebras. We say that $\mathscr{V}$

- has the *amalgamation property* (AP) if, for all $\mathbf{A}, \mathbf{B}_0, \mathbf{B}_1 \in \mathscr{V}$ and all embeddings $f_i\colon \mathbf{A} \to \mathbf{B}_i$, for $i = 0, 1$, there is $\mathbf{C} \in \mathscr{V}$ and embeddings $g_i\colon \mathbf{B}_i \to \mathbf{C}$, $i = 0, 1$, such that $g_0 \circ f_0 = g_1 \circ f_1$,
- is *residually small* (RS) if there is a cardinal κ such that every subdirectly irreducible algebra in $\mathscr{V}$ has cardinality less than κ,
- has the *congruence extension property* (CEP) if, for all $\mathbf{A} \leq \mathbf{B} \in \mathscr{V}$, and congruence α on $\mathbf{A}$, there is $\bar{\alpha} \in \operatorname{Con} \mathbf{B}$ such that $\bar{\alpha} \restriction \mathbf{A} = \alpha$,
- is *n-finite,* for a positive integer n, if every member of $\mathscr{V}$ generated by n elements is finite.

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The least nontrivial congruence on a subdirectly irreducible algebra is called the *monolith*, and the least and greatest congruences on an algebra are denoted 0 and 1 respectively. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}_0 \times \mathbf{A}_1$. It is often convenient to denote the elements of $\mathbf{B}$ as vertical pairs $\binom{x_0}{x_1}$ with $x_i \in A_i$ for $i = 0, 1$. Let $\alpha \in \operatorname{Con} \mathbf{A}_0$. Then there is an induced congruence α_0 on $\mathbf{B}$ given by $\binom{x_0}{x_1}\ \alpha_0\ \binom{y_0}{y_1} \Leftrightarrow x_0\ \alpha\ y_0$. Similarly, for $\beta \in \operatorname{Con} \mathbf{A}_1$, there is a congruence β_1 on $\mathbf{B}$ defined in an analogous way. The particularly important congruences 0_0 and 0_1 are denoted η_0 and η_1 respectively. Given a congruence β on an algebra $\mathbf{D}$, we write $\mathbf{D}(\beta)$ for the subalgebra of $\mathbf{D}^2$ with universe β. That is: $D(\beta) = \left\{ \binom{x}{y} : \ x\ \beta\ y \right\}$.

The commutator is a binary operation, denoted $[\,,\,]$, defined on the congruence lattice of an algebra. In a congruence modular variety $\mathscr{V}$, the commutator has some strong properties. It is *additive*, that is

$$\left[\alpha, \bigvee \Theta\right] = \bigvee_{\theta \in \Theta} [\alpha, \theta] \quad \text{for } \mathbf{A} \in \mathscr{V},\ \alpha \in \operatorname{Con} \mathbf{A}, \text{ and } \Theta \subseteq \operatorname{Con} \mathbf{A},$$

symmetric and *meet-dominated*

$$[\alpha, \beta] = [\beta, \alpha] \subseteq \alpha \wedge \beta \quad \text{for } \mathbf{A} \in \mathscr{V} \text{ and } \alpha, \beta \in \operatorname{Con} \mathbf{A},$$

respects finite direct products

$$\begin{gathered} \text{for } \alpha, \beta \in \operatorname{Con} \mathbf{A} \text{ and } \gamma, \delta \in \operatorname{Con} \mathbf{B} \\ [\alpha_0 \wedge \gamma_1, \beta_0 \wedge \delta_1] = [\alpha, \beta]_0 \wedge [\gamma, \delta]_1 \text{ in } \mathbf{A} \times \mathbf{B}, \end{gathered}$$

and exhibits the following *homomorphism property*

$$\begin{gathered} \text{if } f\colon \mathbf{A} \to \mathbf{B} \text{ is a surjective homomorphism and } \alpha, \beta \in \operatorname{Con} \mathbf{B} \text{ then} \\ f^{-1}[\alpha, \beta] = \left[f^{-1}\alpha, f^{-1}\beta\right] \vee \ker f. \end{gathered}$$

For a systematic development of the subject, we direct the reader to [**6**], especially Chapter 4.

There are three identities involving the commutator that we consider in this paper. We say that an algebra $\mathbf{A}$ has:

$$\begin{array}{lll} \text{C1} \quad \text{if} & \forall \alpha, \beta \in \operatorname{Con} \mathbf{A} & \alpha \wedge [\beta, \beta] = [\alpha \wedge \beta, \beta], \\ \text{C2} \quad \text{if} & \forall \alpha, \beta \in \operatorname{Con} \mathbf{A} & [\alpha, \beta] = \alpha \wedge \beta \wedge [1, 1], \\ \text{R} \quad \text{if} & \forall \mathbf{B} \leq \mathbf{A} & [1_{\mathbf{A}}, 1_{\mathbf{A}}] \restriction \mathbf{B} = [1_{\mathbf{B}}, 1_{\mathbf{B}}]. \end{array}$$

We say that a class $\mathscr{K}$ has one of these properties if and only if every member of $\mathscr{K}$ has the property. Finally for congruences α, β on an algebra $\mathbf{A}$, we define $(\alpha : \beta)$ to be the largest congruence θ such that $[\beta, \theta] \leq \alpha$.

We collect some facts connecting these concepts in the following theorem.

THEOREM 0. *Let $\mathscr{V}$ be a congruence modular variety.*

(1) *$\mathscr{V}$ satisfies* C1 *if and only if, for every subdirectly irreducible algebra $\mathbf{A}$ of $\mathscr{V}$ with monolith β, if $\theta = (0 : \beta)$ then $[\theta, \theta] = 0$.*
(2) *$\mathscr{V}$ satisfies* C2 *if and only if every subdirectly irreducible algebra $\mathbf{A}$ of $\mathscr{V}$ satisfies* C2, *and this in turn is equivalent to $\mathbf{A}$ being either abelian or prime.*
(3) C2 $\Rightarrow$ C1.

(4) *If* $\mathscr{V}$ *is a residually small variety then it satisfies* C1. *Conversely, if* $\mathscr{V}$ *is finitely generated and satisfies* C1, *then it is residually small.*

(5) *Every variety with CEP satisfies* $\mathrm{C2}+R$.

(1) and (4) are from [**5**], (2) and (5) are from [**7**]. (**A** is *prime* iff $\alpha, \beta \neq 0 \Rightarrow [\alpha, \beta] \neq 0$.) (3) follows easily from the definitions.

Our first objective is the following

THEOREM 1. *Let* $\mathscr{V}$ *be a congruence modular variety. If* $\mathscr{V}$ *has* AP *and* RS, *then* $\mathscr{V}$ *has* C2.

We require a pair of lemmas. The first of these is virtually identical to Theorem 10.9 of [**6**]. We include the proof for completeness.

LEMMA 2. *Let* $\mathscr{V}$ *be a congruence modular, residually small variety. If* $\mathscr{V}$ *fails* C2, *then* $\mathscr{V}$ *contains a subdirectly irreducible algebra* $\mathbf{A}$ *with monolith* β, *and an endomorphism* f *such that*

(1) $0=[\beta, \beta]<\beta=[\beta, 1]$

(2) $f=f^2$

(3) $x \,\beta\, y \quad \Leftrightarrow \quad f(x)=f(y) \,\beta\, y.$

PROOF. By Theorem 0, $\mathscr{V}$ satisfies C1 since it is residually small. If $\mathscr{V}$ fails C2, there is a subdirectly irreducible algebra $\mathbf{D}$ which is neither abelian nor prime. Let γ be the monolith of $\mathbf{D}$ and $\kappa=(0:\gamma)$. Then, since $\mathbf{D}$ is not prime, $[\gamma, \gamma]=0$ and since $\mathbf{D}$ is not abelian, but satisfies C1, $\gamma \leq \kappa<1$ and $[\kappa, \kappa]=0$.

Let $\Delta=\Delta_{\gamma,\kappa}$ be the congruence on $\mathbf{D}(\gamma)$ generated by

$$\left\{\left\langle\binom{x}{x},\binom{y}{y}\right\rangle : x \,\kappa\, y\right\}.$$

Observe that on $\mathbf{D}(\gamma)$, $\kappa_i=(\eta_i:\gamma_i)$, for $i=0,1$, since

$$[\delta, \gamma_i] \leq \eta_i \Rightarrow [\delta \vee \eta_i, \gamma_i]=[\delta, \gamma_i] \vee [\eta_i, \gamma_i] \leq \eta_i \Rightarrow \delta \leq \delta \vee \eta_i \leq \kappa_i$$

by the homomorphism property of the commutator. Furthermore the following relationships hold among the congruences of $\mathbf{D}(\gamma)$:

$$\begin{aligned} \eta_i &\prec \gamma_i \leq \kappa_i \\ \gamma_0 &= \gamma_1=\eta_0 \vee \eta_1 \\ \Delta \vee \eta_i &= \kappa_0=\kappa_1 \\ \Delta \wedge \eta_i &= 0 \end{aligned} \qquad \text{for } i=0,1.$$

The last of these follows from the fact that $[\gamma, \kappa]=0$. See [**6**, Theorem 4.9]. It follows from these identities that $\gamma_0/\eta_0 \searrow \eta_1/0 \nearrow \kappa_0/\Delta \searrow \eta_0/0 \nearrow \gamma_1/\eta_1$, and therefore, η_0 and η_1 are atoms of $\operatorname{Con}\mathbf{D}(\gamma)$.

We claim that Δ is a completely meet-irreducible congruence with (unique) cover κ_0. Let $\lambda>\Delta$ and $\lambda \neq \kappa_0$. From the computation above, κ_0 covers Δ, so $\lambda \not\leq \kappa_0$. Since $\kappa_0=(\eta_0:\gamma_0)$ we have $[\lambda, \eta_0 \vee \eta_1]=[\lambda, \gamma_0] \not\leq \eta_0$ and therefore $[\lambda, \eta_1] \not\leq \eta_0$. It follows that $\lambda \wedge \eta_1 \neq 0$. But η_1 is an atom, so $\lambda \geq \eta_1$, thus $\lambda \geq \eta_1 \vee \Delta=\kappa_0$ as desired.

Define $\mathbf{A}$ to be $\mathbf{D}(\gamma)/\Delta$. Then $\mathbf{A}$ is subdirectly irreducible with monolith $\beta=\kappa_0/\Delta$. Note that $[\kappa_0, \kappa_0]=[\eta_0 \vee \Delta, \eta_1 \vee \Delta] \leq \Delta$, so $[\beta, \beta]=0$ on $\mathbf{A}$. Also $\mathbf{A}$ is nonabelian. For if it were abelian, then in $\mathbf{D}(\gamma)$, $[\eta_1, 1] \leq \Delta \wedge \eta_1 = 0$. But then $[\gamma_0, 1] = [\eta_0 \vee \eta_1, 1] \leq \eta_0$ which contradicts the fact that $(\eta_0 : \gamma_0) = \kappa_0 < 1$. Therefore, in $\mathbf{A}$, $[\beta, 1] = \beta$, by C1, verifying (1) of the lemma.

We define f: $\mathbf{A} \to \mathbf{A}$ by $f\left(\binom{x}{y}/\Delta\right) = \binom{x}{x}/\Delta$. As $\Delta \leq \kappa_0$, this is well defined. That f is an endomorphism satisfying conditions (2) and (3) above is a straightforward verification. □

In order to continue, we need to recall some facts about our modular variety $\mathscr{V}$. There is a ternary term d in the language of $\mathscr{V}$, called the *difference term*, with the following properties:

(1) $\mathscr{V} \models d(x,x,y) = y$.
(2) For every $\mathbf{B}$ in $\mathscr{V}$, abelian congruence θ on $\mathbf{B}$, and $b \in B$, $\langle b/\theta, d\rangle$ is a ternary group, denoted $M(\theta, b)$, and for each n-ary term function t, if $t(b_1, b_2, \ldots, b_n) = c$ then t is a ternary group homomorphism from $M(\theta, b_1) \times \cdots \times M(\theta, b_n)$ to $M(\theta, c)$.

For the appropriate definitions and proofs see [**6**, 5.5–5.8].

LEMMA 3. *Let* $\mathbf{A}$ *be an algebra satisfying the conclusions of Lemma* 2. *There are automorphisms* e_0 *and* e_1 *of* $\mathbf{A}(\beta)$ *given by*:

$$e_0\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d(x, fx, y) \\ y \end{pmatrix} \qquad \text{and} \qquad e_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ d(x, fy, y) \end{pmatrix}$$

where d *is the difference term for* $\mathscr{V}$.

PROOF. Recall that for any $x, y \in A$, $\binom{x}{y} \in A(\beta) \Leftrightarrow x \mathrel{\beta} y \Leftrightarrow f(x) = f(y) \mathrel{\beta} y$ by (3) of Lemma 2. Suppose $e_0\binom{x}{y} = e_0\binom{u}{v}$. Then $y = v$ and therefore $x \mathrel{\beta} u$, so $f(x) = f(u)$ and all six elements lie in the same ternary group, $M(\beta, x)$. We have $x - f(x) + y = d(x, f(x), y) = d(u, f(u), v) = u - f(x) + y$, so $x = u$. Thus e_0 is injective. Similarly, one can check that

$$e_0^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d(x, y, fx) \\ y \end{pmatrix}.$$

That e_0 is a homomorphism follows from the fact that every term, in particular every basic operation, is a ternary group homomorphism on the $M(\beta, x)$-blocks. □

PROOF OF THEOREM 1. Assume that $\mathscr{V}$ has AP and RS, but fails C2. We shall derive a contradiction. Let $\mathbf{A}$, β, e_0 and e_1 be as in Lemmas 2 and 3. Since $\mathscr{V}$ is residually small, there is a maximal essential extension, $\mathbf{E}$, of $\mathbf{A}$ in $\mathscr{V}$ (see [**12**]). Observe that $\mathbf{E}$ is subdirectly irreducible. Call its monolith μ. Without loss of generality, we may assume that $\mathbf{A} \subseteq \mathbf{E}$ and $\mu \restriction A \supseteq \beta$.

The automorphisms e_0 and e_1 of Lemma 3 are also embeddings of $\mathbf{A}(\beta)$ into $\mathbf{E}^2$. Let us also define $e_2 : \mathbf{A}(\beta) \to \mathbf{E}^2$ to be the identity map. By the amalgamation property (applied twice), there is an algebra $\mathbf{Q}$ in $\mathscr{V}$ and maps $s_j : \mathbf{E}^2 \to \mathbf{Q}$, for $j = 0, 1, 2$, such that $s_0 \circ e_0 = s_1 \circ e_1 = s_2 \circ e_2$. Furthermore, there is a map $r : \mathbf{E} \to \mathbf{E}^2$ given by $r(x) = \binom{x}{x}$. Then $r \restriction A : \mathbf{A} \to \mathbf{A}(\beta)$, and Figure 1 commutes.

Since $\mathbf{E}$ is a maximal essential extension of $\mathbf{A}$, it is an *absolute retract* in $\mathscr{V}$, that is, a retract of each of its $\mathscr{V}$-extensions. Therefore there is a retraction $u : \mathbf{Q} \to \mathbf{E}$ such that $u \circ s_2 \circ r = \mathrm{id}_{\mathbf{E}}$. Define

$$\rho_i = \ker(u \circ s_i) \in \operatorname{Con} \mathbf{E}^2 \qquad \text{for } i = 0, 1, 2.$$

$$
\begin{tikzcd}
 & \mathbf{Q} \arrow[ddrr, bend left, "u"] & & \\
\mathbf{E}^2 \arrow[ur, "s_0"] & \mathbf{E}^2 \arrow[u, "s_1"'] & \mathbf{E}^2 \arrow[ul, "s_2"'] & \\
 & \mathbf{A}(\beta) \arrow[ul, "e_0"] \arrow[u, "e_1"'] \arrow[ur, "e_2"'] & & \mathbf{E} \arrow[ul, "r"'] \\
 & & \mathbf{A} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

FIGURE 1

CLAIM. *For each i, if $\rho_i \neq 0$, then either*

$$\rho_i \geq \eta_0 \wedge \mu_1 \text{ and } \rho_i \wedge \eta_1 = 0 \text{ or}$$
$$\rho_i \geq \eta_1 \wedge \mu_0 \text{ and } \rho_i \wedge \eta_0 = 0.$$

PROOF OF CLAIM. Let us write ρ for ρ_i in the proof of the claim. For any nonzero congruence α on $\mathbf{E}$, $[1_{\mathbf{E}}, \alpha] \supseteq [1_{\mathbf{E}}, \mu] \supseteq [1_{\mathbf{A}}, \beta] = \beta \neq 0_{\mathbf{A}}$; that is, $\mathbf{E}$ is centerless. Therefore, by the homomorphism property, $\mathbf{E}^2$ is centerless. Thus

$$0_{\mathbf{E}^2} \neq [1, \rho] = [\eta_0 \vee \eta_1, \rho] = [\eta_0, \rho] \vee [\eta_1, \rho]$$

so, say $[\eta_0, \rho] \neq 0$. Then $\eta_0 \wedge \rho \neq 0$, implying $\rho \geq \eta_0 \wedge \mu_1$ (since $\eta_0/0 \nearrow 1/\eta_1$). Furthermore, $\rho \wedge \eta_1 = 0$, for if not, then $\rho \geq \eta_1 \wedge \mu_0$, so $\rho \geq (\eta_0 \wedge \mu_1) \vee (\eta_1 \wedge \mu_0) = \mu_0 \wedge \mu_1$. But choose $(a, b) \in \beta - 0_{\mathbf{A}}$. Recall $\mu \supseteq \beta$ and $f(a) = f(b)$. Since d is a term, it follows that $d(a, fa, a)\ \mu\ d(b, fb, b)$ and therefore $e_i\binom{a}{a} \equiv e_i\binom{b}{b}$ (mod $\mu_0 \wedge \mu_1$), hence modulo ρ as well. But then $u \circ s_i \circ e_i\binom{a}{a} = u \circ s_i \circ e_i\binom{b}{b}$, so by the commutativity of Figure 1, $a = u \circ s_2 \circ r(a) = u \circ s_2 \circ r(b) = b$, which is a contradiction. Therefore we must have $\rho \ngeq \eta_1 \wedge \mu_0$, so $\rho \wedge \eta_1 = 0$. This proves the claim.

We first apply the claim to ρ_2. Certainly, $\rho_2 \neq 0$, in fact, $\binom{x}{y}\ \rho_2\ \binom{z}{z}$, for $z = u \circ s_2\binom{x}{y}$ (since $u \circ s_2\binom{z}{z} = u \circ s_2 \circ r(z) = z$.) Let us assume that $\rho_2 \geq \eta_0 \wedge \mu_1$, and we will derive a contradiction. The case $\rho_2 \geq \eta_1 \wedge \mu_0$ will follow by symmetry. We have $[1, \rho_2] = [\eta_0 \vee \eta_1, \rho_2] \leq \eta_0 \vee (\eta_1 \wedge \rho_2) = \eta_0$, so $[1, \rho_2 \vee \eta_0] \leq \eta_0$ implying $\rho_2 \leq \eta_0$ since $\mathbf{E}$ is centerless. For any $x, y \in E$, $\binom{x}{y} \equiv r \circ u \circ s_2\binom{x}{y} = \binom{z}{z}$ (mod ρ_2) for some $z \in E$, whence $x = z$. Thus $\binom{x}{y}\ \rho_2\ \binom{x}{x}\ \rho_2\ \binom{x}{y'}$ and we conclude that $\rho_2 = \eta_0$.

Therefore

$$\eta_0^{\mathbf{A}(\beta)} = e_2^{-1}(\rho_2) = \ker(u \circ s_2 \circ e_2) = \ker(u \circ s_0 \circ e_0) = e_0^{-1}(\rho_0).$$

Observe that for every $a \in A$, $e_0\binom{fa}{a} = \binom{d(fa,fa,a)}{a} = \binom{a}{a}$. Choose $(a,b) \in \beta - 0_{\mathbf{A}}$. Then

$$\begin{pmatrix} fa \\ a \end{pmatrix} \eta_0 \begin{pmatrix} fb \\ b \end{pmatrix} \Rightarrow u \circ s_0 \circ e_0 \begin{pmatrix} fa \\ a \end{pmatrix} = u \circ s_0 \circ e_0 \begin{pmatrix} fb \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ a \end{pmatrix} \rho_0 \begin{pmatrix} b \\ b \end{pmatrix}.$$

Now apply the claim to ρ_0. If $\rho_0 \geq (\eta_0 \wedge \mu_1)$ then $\binom{a}{b}\ \rho_0\ \binom{a}{a}\ \rho_0\ \binom{b}{b}$, contradicting $\rho_0 \wedge \eta_1 = 0$. If $\rho_0 \geq (\eta_1 \wedge \mu_0)$ then

$$\eta_0 = e_0^{-1}(\rho_0) \geq e_0^{-1}(\eta_1 \wedge \mu_0) \geq \eta_1 \wedge \beta_0$$

on $\mathbf{A}(\beta)$ which is false, proving the theorem. □

We now turn to our second theorem. In [**1**, Theorem 6] it was proved that if $\mathscr{V}$ is congruence modular, 4-finite and satisfies C2 and R, then $\mathscr{V}$ satisfies AP + RS $\Rightarrow$ CEP. Theorem 1 above eliminates the need to assume C2. The proof below does without the assumption of 4-finiteness. In addition, it provides a considerable simplification of the previous argument. We require one lemma from that paper.

LEMMA. *Let $\mathscr{V}$ be a congruence modular variety satisfying AP and RS. Let $\mathbf{A}$ be a subdirectly irreducible member of $\mathscr{V}$, and assume $\mathbf{A}$ is an essential extension of $\mathbf{B}_0 \times \mathbf{B}_1$. Then either $\mathbf{B}_0$ or $\mathbf{B}_1$ is trivial.*

THEOREM 4. *Let $\mathscr{V}$ be congruence modular and satisfy R. If $\mathscr{V}$ has AP and RS, then $\mathscr{V}$ has CEP.*

PROOF. By Theorem 1, $\mathscr{V}$ has C2. Let $\mathbf{A}, \mathbf{B} \in \mathscr{V}$, with $\mathbf{A} \leq \mathbf{B}$ and θ a completely meet-irreducible congruence on $\mathbf{A}$. It suffices to show that θ extends to $\mathbf{B}$. By R, $[1_{\mathbf{B}}, 1_{\mathbf{B}}] \restriction \mathbf{A} = [1_{\mathbf{A}}, 1_{\mathbf{A}}]$. Thus $\mathbf{A}/[1, 1]$ can be embedded in $\mathbf{B}/[1, 1]$. Suppose first that $\theta \geq [1, 1]$ on $\mathbf{A}$. Since $\mathbf{B}/[1, 1]$ is abelian, it has CEP, so the congruence $\theta/[1, 1]$ (of $\mathbf{A}/[1, 1]$) extends to a congruence $\psi/[1, 1]$ on $\mathbf{B}/[1, 1]$. Then $\psi \in \operatorname{Con} \mathbf{B}$ and $\psi \restriction \mathbf{A} = \theta$.

So we may assume that $\theta \not\geq [1, 1]$. Let f be the embedding of $\mathbf{A}$ into $\mathbf{B}$. Define $\mathbf{S}$ to be $(\mathbf{A}/\theta) \times \mathbf{A}$ and $g\colon \mathbf{A} \to \mathbf{S}$ by $g(a) = \binom{a/\theta}{a}$. We apply the amalgamation property to $(\mathbf{A}, \mathbf{B}, \mathbf{S}, f, g)$ yielding $(\mathbf{D}, \bar{f}, \bar{g})$ with $\mathbf{D} \in \mathscr{V}$ (Figure 2).

$$\begin{tikzcd}
 & \mathbf{D} & \\
\mathbf{B} \arrow[ur, "\bar{f}"] & & \mathbf{S} \arrow[ul, "\bar{g}"'] \\
 & \mathbf{A} \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

FIGURE 2

Let η_0 and η_1 denote the coordinate projection kernels on $\mathbf{S}$. It suffices to find γ in $\operatorname{Con}\mathbf{D}$ such that $\bar{g}^{-1}(\gamma) = \eta_0$. By the assumptions on θ, $\mathbf{A}/\theta$ is subdirectly irreducible with monolith μ. Thus, there is a congruence μ_0 on $\mathbf{S}$ covering η_0. The commutator respects finite direct products, so $[1_{\mathbf{S}}, 1_{\mathbf{S}}] = [1, 1]_0 \wedge [1, 1]_1 \geq \mu_0 \wedge \eta_1$ since $\mathbf{A}/\theta$ is nonabelian. Therefore

$$(*) \qquad \text{for all } \beta \in \operatorname{Con}\mathbf{S} \qquad \beta \nleq \eta_0 \Leftrightarrow \beta \geq \mu_0 \wedge \eta_1.$$

PROOF. Certainly, $\eta_0 \geq \beta \geq \mu_0 \wedge \eta_1$ implies $\mu_0 \wedge \eta_1 = \mu_0 \wedge \eta_1 \wedge \eta_0 = 0$, which is false. Conversely, if $\beta \nleq \eta_0$ then $\beta \vee \eta_0 \geq \mu_0$. Therefore by additivity and C2,

$$\beta \geq [\beta \vee \eta_0, \beta \vee \eta_1] = (\beta \vee \eta_0) \wedge (\beta \vee \eta_1) \wedge [1, 1] \geq \mu_0 \wedge \eta_1.$$

Now let γ be a maximal congruence on $\mathbf{D}$ such that $\bar{g}^{-1}(\gamma) \leq \eta_0^{\mathbf{S}}$. Then γ is completely meet-irreducible. For suppose $\gamma = \bigcap_{j\in J} \gamma_j$ with $\gamma_j > \gamma$, all j. Then $\bar{g}^{-1}(\gamma_j) \nleq \eta_0$, so by $(*)$, $\bar{g}^{-1}(\gamma_j) \geq \mu_0 \wedge \eta_1$, for all $j \in J$, and therefore

$$\bar{g}^{-1}(\gamma) = \bigcap_{j\in J} \bar{g}^{-1}(\gamma_j) \geq \mu_0 \wedge \eta_1$$

which implies

$$\bar{g}^{-1}(\gamma) \nleq \eta_0,$$

which is a contradiction.

Therefore $\mathbf{D}/\gamma$ is subdirectly irreducible and $\bar{g}^{-1}(\gamma) = \eta_0 \wedge \delta_1$ for some $\delta \in \operatorname{Con}\mathbf{A}$, by modularity. Then $\mathbf{S}/\bar{g}^{-1}(\gamma) \cong \mathbf{A}/\theta \times \mathbf{A}/\delta$ and we have an induced embedding $\bar{g}/\gamma : \mathbf{A}/\theta \times \mathbf{A}/\delta \to \mathbf{D}/\gamma$. Furthermore, by the maximality of γ, this embedding is essential. Therefore, by the lemma mentioned above, either $\mathbf{A}/\theta$ or $\mathbf{A}/\delta$ is trivial. But $\theta \ngeq [1, 1]$, so we must have $\mathbf{A}/\delta$ trivial, which means $\delta = 1_{\mathbf{A}}$, and therefore $\bar{g}^{-1}(\gamma) = \eta_0$ as desired. □

COROLLARY 5. *Let $\mathscr{V}$ be a congruence distributive variety. If $\mathscr{V}$ has AP and RS, then $\mathscr{V}$ has CEP.*

COROLLARY 6. *Let $\mathscr{V}$ be a congruence modular variety that is 2-finite. If $\mathscr{V}$ has AP and RS, then $\mathscr{V}$ has CEP.*[1]

PROOF. If $\mathscr{V}$ is congruence distributive, the commutator reduces to intersection. Thus $\mathscr{V}$ trivially has R and the result follows from Theorem 4. Suppose $\mathscr{V}$ is congruence modular and 2-finite. By Theorem 1, $\mathscr{V}$ has C2. By [**2**, Theorem 8] $\mathscr{V}$ has R. Therefore, by Theorem 4, $\mathscr{V}$ has CEP. □

It is well known that an arbitrary variety has enough injectives if and only if it has AP, RS, and CEP. By the above corollaries, it follows that any congruence modular, 2-finite variety has enough injectives if and only if it has AP and RS.

As applications we offer the following corollaries.

COROLLARY 7. *Let $\mathscr{V}$ be a variety of groups with AP and RS. Then $\mathscr{V}$ is abelian.*

PROOF. By Theorem 1, $\mathscr{V}$ satisfies C2. It suffices to show that every subdirectly irreducible group in $\mathscr{V}$ is abelian. Let $\mathbf{G}$ be subdirectly irreducible, and let

[1] ADDED IN PROOF. Recently, Keith Kearnes has shown that the assumption in Corollary 6 that $\mathscr{V}$ be 2-finite can be dropped. Thus, any congruence modular variety with AP and RS has CEP.

$\mathbf{M}$ be the normal subgroup corresponding to the monolith. Choose any $a \in M - \{1\}$, and let $\mathbf{A}$ be the subgroup generated by a. Then $\mathbf{A}$ is abelian and $\mathbf{G}$ is an essential extension of $\mathbf{A}$. Therefore, by [**2**, Theorem 7], $\mathbf{G}$ is abelian. □

REMARKS. In [**10**, p. 43] H. Neumann states the following theorem: If $\mathscr{V}$ is a variety of groups with AP, then either every *finite* member of $\mathscr{V}$ is abelian, or else $\mathscr{V}$ is the variety of all groups.

Observe that the argument in Corollary 7 generalizes to any congruence modular variety $\mathscr{V}$ such that

(i) the free algebra on 2 generators is abelian or
(ii) the free algebra on one generator is nontrivial, abelian and has an idempotent element.

COROLLARY 8. *The variety of all squags is not residually small.*

PROOF. A squag is a groupoid $\langle S, \cdot \rangle$ satisfying

$$x \cdot x = x, \qquad x \cdot y = y \cdot x, \qquad x \cdot (x \cdot y) = y.$$

Every squag is a quasigroup ("squag" is short for "Steiner quasigroup"), so it follows that the variety of squags is congruence modular.

It is implicit in Bruck [**3**, Theorem 2.1] that the variety of squags has AP. On the other hand, Quackenbush [**11**, 7.6] showed that this variety does not have CEP. It is easy to see that the variety is 2-finite, in fact the free squag on $\{x, y\}$ has 3 elements: $x, y, x \cdot y$. Therefore, by Corollary 6, the variety of squags is not residually small. □

A variety $\mathscr{V}$ is *directly representable* if it is finitely generated and contains only finitely many finite, directly indecomposable algebras.

COROLLARY 9. (1) *Let $\mathscr{V}$ be a directly representable variety with AP. Then $\mathscr{V}$ has CEP. In fact, $\mathscr{V}$ is a varietal product $\mathscr{A} \otimes \mathscr{D}$, in which $\mathscr{A}$ is abelian and $\mathscr{D}$ is a discriminator variety.*

(2) *Let $\mathscr{V}$ be a congruence modular, semi-simple, finitely generated variety with AP. Then $\mathscr{V} = \mathscr{A} \otimes \mathscr{D}$ in which $\mathscr{A}$ is abelian and $\mathscr{D}$ is filtral.*

PROOF. By [**8**], every directly representable variety is congruence permutable (hence modular), satisfies C2 and every subdirectly irreducible member is either simple or abelian. In case (2), semisimplicity implies C2. Now let $\mathscr{V}$ represent the variety in either case. $\mathscr{V}$ is finitely generated, hence 2-finite. $\mathscr{V}$ has C2, hence C1, hence is residually small (Theorem 0). Therefore, by Corollary 6, $\mathscr{V}$ has CEP.

Let $\mathscr{A}$ be the subvariety of $\mathscr{V}$ consisting of all abelian algebras, and let $\mathscr{D}$ be the subvariety generated by all simple, non-abelian algebras. Then by [**7**, 7.2], $\mathscr{V} = \mathscr{A} \otimes \mathscr{D}$, and $\mathscr{D}$ is a congruence distributive variety, semisimple and with CEP. Therefore $\mathscr{D}$ is filtral, proving (2). In case (1), $\mathscr{D}$ is congruence permutable as well, so it is a discriminator variety. □

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DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA 50011

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720