A SPECIAL CASE OF POSITIVITY

S. P. DUTTA

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ABSTRACT. In this note we prove a special case of positivity of Serre’s Conjecture on intersection multiplicity of modules [S]. The conjecture can be stated as follows.

Let \( R \) be a regular local ring and let \( M \) and \( N \) be two finitely generated modules over \( R \) such that \( l(M \otimes N) < \infty \). Then \( \chi(M, N) = \sum_{i=0}^{\dim R} (-1)^i l(\text{Tor}_i^R(M, N)) \geq 0 \), the sign of inequality holds if and only if \( \dim M + \dim N = \dim R \).

Serre proved the conjecture in the equicharacteristic and in the unramified case. Recently P. Roberts [R] and H. Gillet and C. Soulé [H-G] proved independently the vanishing part, i.e. \( \chi(M, N) = 0 \) when \( \dim M + \dim N < \dim R \) in more generality. The positivity part, i.e. \( \chi(M, N) > 0 \) when \( \dim M + \dim N = \dim R \) is still very much an open question.

We write \( R = V[[x_1, \ldots, x_n]]/(f) \), \( V \) a complete discrete valuation ring, \( p \) a generator of the maximal ideal of \( V \), \( p \in m^2 \) where \( m \) is the maximal ideal of \( R \) and \( f \in m - m^2 \). We divide the whole problem into three parts:

1. \( pM = 0, pN = 0 \). This case was proved by Malliavin-Brameret [M].
2. \( p \) is a non-zero-divisor on \( M \) and \( p \) is nilpotent on \( N \).
3. \( p \) is a non-zero-divisor on both \( M \) and \( N \).

The theorem which we are going to prove is the following

THEOREM. Let \( R \) be a regular local ring. Let \( M \) and \( N \) be two finitely generated modules over \( R \) such that

(i) \( M \) is Cohen-Macaulay.
(ii) \( l(M \otimes N) < \infty \) and \( \dim M + \dim N = \dim R \).
(iii) \( p^tN = 0 \) for some integer \( t \) and \( p \) is a non-zero-divisor on \( M \).

Then \( \chi(M, N) > 0 \).

The above theorem was already proved by the author in the case when \( \dim R \leq 5 \) in [D2]. The vanishing theorem of Roberts (Gillet and Soulé) and the techniques developed by the author in [D1] now make it possible to prove the above version.

PROOF OF THE THEOREM. We divide the proof into two steps.

Step 1. Let \( R \) be a Gorenstein local ring of characteristic \( p > 0 \). Let \( M \) be a module of finite projective dimension and let \( N \) be any other module over \( R \) such that \( l(M \otimes N) < \infty \) and \( \dim M + \dim N \leq \dim R \).

Let \( f : R \to R \) be the Frobenius map, i.e. \( f(x) = x^p \). We denote by \( f^R \) the bialgebra \( R \), having the structure of an \( R \)-algebra from the left by \( f^R \) and from...
the right by the identity map, i.e., if $\alpha \in R$, $x \in f^n_R$, $\alpha x = \alpha^{p^n} x$, and $x\alpha = x\alpha$.
We assume $K = R/m$, where $m$ is the maximal ideal of $R$, is perfect. (This assumption is not at all restrictive with respect to generalized type of intersection multiplicity conjectures.) We denote by $F^n(M)$ the object $M \otimes f^n_R$. We define $\chi_{\infty}(M, N) = \lim_{n \to \infty} \chi(F^n(M), N)/p^{n \cdot \text{codim} M}$. The following properties of $\chi_{\infty}$ were proved in [D1].

1. If $\dim M + \dim N < \dim R$, then $\chi_{\infty}(M, N) = 0$ (Corollary 1, p. 437).
2. If $M$ is Cohen-Macaulay, then
   $$\chi_{\infty}(M, N) = \lim_{n \to \infty} l(F^n(M) \otimes N)/p^{n \cdot \text{codim} M}$$
   and this is a positive integer if $R$ is a complete intersection.
3. $\chi_{\infty}(M, N) = \chi(M, N)$ if the vanishing conjecture holds for every pair $(M, T)$ with $l(M \otimes T) < \infty$ and $\dim M + \dim T < \dim R$ (this assertion follows easily from Proposition 1.2 of [D1]).

**Step 2.** Under the hypothesis in our theorem, since $\chi$ is additive, we can assume $pN = 0$. Since $p$ is a non-zero-divisor on both $R$ and $M$ and $pN = 0$, we have
   (i) $\chi^R(M, N) = \chi^{R/pR}(M/pM, N)$,
   (ii) $\text{P.d.}_{R/pR} M/pM < \infty$ and $\chi^{R/pR}(M/pM, T) = 0$, where
   $$\dim M/pM + \dim T < \dim R/pR$$
   (since this implies $\dim M + \dim T < \dim R$, and $\chi(M, T) = 0$ [G-S, R]).
   (iii) $M/pM$ is Cohen-Macaulay over $R/pR$ with $\text{Ch. } R/pR = p > 0$.

We denote $R/pR$ by $\overline{R}$. We have by (ii) and (3) of Step 1
   $$\chi_{\infty}(M, N) = \chi^R(M, N) = \chi^R(M, N).$$
Moreover by (iii) and (2) of Step 1
   $$\chi_{\infty}(M, N) = \lim_{n \to \infty} l(F^n(M) \otimes N)/p^{n \cdot \text{codim} M}.$$
Hence $\chi^R(M, N) > 0$.

**REMARK.** Unfortunately, $\chi_{\infty}$ fails to behave like a “multiplicity function” over $R$ for pairs $(M, N)$ with $\text{P.d. } M < \infty$, $l(M \otimes N) < \infty$, $\dim M + \dim N = \dim R$ when $M$ is not Cohen-Macaulay. This was pointed out in [D-H-M]. The counterexample discussed there gives rise to a module $M$ with $\text{P.d. } M < \infty$, $\dim M = 1$, depth $M = 0$ and another module $N$ such that $\chi_{\infty}(M, N)$ is negative.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19104-6395

Current address: Department of Mathematics, University of Illinois, Urbana, Illinois 61801

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