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SPOTS OVER ANALYTICALLY NORMAL LOCAL DOMAINS

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ABSTRACT. A result of Lipman on 2-dimensional analytically normal local domains is extended to dimension $d > 2$, thereby enlarging the class of local domains known to have certain finiteness properties with respect to birational extensions.

This note is an addendum to the paper, *A criterion for spots* [**HHS**]. Its purpose is to extend a result of Lipman [**L**] to dimension $d > 2$, and thereby enlarge the class of d-dimensional local domains known to have the property that every d-dimensional quasi-unmixed local domain birationally dominating them is a spot over them. (Recall that a local ring S is said to be a spot over a ring R if S is the localization at a prime ideal of a finitely generated ring over R.)

In [**HHS**] the following property is defined.

DEFINITION. A d-dimensional local domain (R, m) has property L, respectively $\overline{L}$, if every d-dimensional normal spot birationally dominating R is analytically irreducible, respectively analytically normal. In [**HHS**] it is proved that if (R, m) is a d-dimensional analytically unramified local domain with property L, then every d-dimensional normal or quasi-unmixed local domain which birationally dominates R is a spot over R.

The class of local domains with $\overline{L}$ includes all one dimensional local domains, all excellent local domains and, by a result of Lipman [**L**], all 2-dimensional analytically normal domains. It is an open question whether all d-dimensional analytically normal local domains, for $d > 2$, are in this class. The following result extends Lipman's result to certain d-dimensional analytically normal local domains.

PROPOSITION. *Let (R, m) be a d-dimensional analytically normal local domain such that R/p is Nagata for all height 2 primes p of R. Then R has $\overline{L}$. In fact, every normal spot which birationally dominates R is analytically normal.*

PROOF. Let (S, n) be a normal local domain birationally dominating R which is a spot over R. We will show that S is analytically normal. With $\widehat{\ }$ denoting maximal ideal-adic completion, we have

$$R \subset \hat{R} \subset \hat{R} \otimes_R S \subset (\hat{R} \otimes_R S)_{n(\hat{R} \otimes_R S)} = T.$$

Now

$$\hat{R} \otimes_R S/n^i(\hat{R} \otimes_R S) \cong \hat{R} \otimes_R (S/n^i S) \cong R/m^i R \otimes_{R/m^i R} S/n^i S \cong S/n^i S,$$

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so $\hat{T} = \hat{S}$. Since T is a spot over $\hat{R}$, T is excellent. Thus to prove that S is analytically normal it is sufficient to prove that T is normal.

T is flat over S, so for any prime ideal P of T

$$(*) \qquad \operatorname{depth} T_P = \operatorname{depth} S_Q + \operatorname{depth} T_P/QT_P,$$

where $Q = P \cap S$. We will prove that T_P is normal for all primes P of T which are maximal primes of principal ideals of T [**K**, Theorem 53]. Let P be one such. Let $p = P \cap R$ and $Q = P \cap S$. Suppose first that $\operatorname{ht} p \leq 1$. Then $R_p = S_Q$ and T_P is a localization of $\hat{R} \otimes_R R_p \cong \hat{R}_{R-p}$ which is normal since $\hat{R}$ is normal.

Suppose now that $\operatorname{ht} p > 1$. Since P is a maximal prime of a principal ideal of T, depth $T_P = 1$. It follows from $(*)$ that $\operatorname{ht} Q = 1$ because S is normal. Now $R/p \subset S/Q$ so S/Q is a domain which is a spot over the Nagata domain R/p. Consequently, S/Q is analytically unramified, i.e., $Q\hat{S}$ is a radical idea. Since $P \supseteq QT = Q\hat{T} \cap T = Q\hat{S} \cap T$, it follows that $T/QT \subset \hat{S}/Q\hat{S}$. Hence QT is a radical ideal and T_P/QT_P is reduced. Now S_Q is a DVR so $QS_Q = xS_Q$, for some $x \in Q$. Thus $PT_P = QT_P = (QS_Q)T_P = xT_P$, and T_P is also a DVR.

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