AN INEQUALITY FOR THE INTEGRAL MEANS OF A HADAMARD PRODUCT
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ABSTRACT. Motivated by Colzani's paper [1] we prove that
\[ M_q(r, f \ast g) \leq (1 - r)^{1-1/p}\|f\|_p \|g\|_q, \quad 0 < r < 1, \]
where \(0 < p < 1, p \leq q \leq \infty\) and \(f \ast g\) is the Hadamard product of \(f \in H^p\) and \(g \in H^q\).

For a function \(F\), continuous in the disc \(U = \{z: |z| < 1\}\) let
\[ M_p(r, F) = \frac{1}{2\pi} \int_0^{2\pi} |F(re^{it})|^p \, dt, \quad 0 < p < \infty, \]
\[ M_\infty(r, F) = \max_{0 \leq t < 2\pi} |F(re^{it})|, \]
where \(0 \leq r < 1\). The Hardy class \(H^p\) consists of those \(f\) which are analytic in \(U\) and satisfy the condition
\[ \|f\|_p := \sup_{0 < r < 1} M_p(r, f) < \infty. \]

If \(f(z) = \sum a_n z^n\) and \(g(z) = \sum b_n z^n\) are analytic in \(U\), then their Hadamard product \(f \ast g\), defined by
\[ (f \ast g)(z) = \sum_{n=0}^\infty a_n b_n z^n, \]
is analytic in \(U\). It is well known that if \(f \in H^1\) and \(g \in H^q, q \geq 1\), then \(M_q(r, f \ast g) \leq \|f\|_1 \|g\|_q\) (\(0 < r < 1\)) and consequently \(f \ast g \in H^q\). This fact is generalized by the following theorem.

**THEOREM.** Let \(f \in H^p\) and \(g \in H^q\), where \(0 < p \leq 1\) and \(p \leq q < \infty\). Then
\[ M_q(r, f \ast g) = O((1 - r)^{1-1/p}), \quad r \to 1^-, \]
and,
\[ M_q(r, f \ast g) \leq (1 - r)^{1-1/p}\|f\|_p \|g\|_q, \quad 0 < r < 1. \]

For the proof we need a familiar lemma.

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LEMMA. If $F \in H^p$, $0 < p \leq 1$, then
\[ M_1(r, F) \leq (1 - r^2)^{1 - 1/p} \|F\|_p, \quad 0 < r < 1. \]

PROOF. Since
\[ M_1(r, F) \leq M_{\infty}^{1-p}(r, F)M_p^p(r, F), \]
we have to prove that
\[ M_{\infty}(r, F) \leq (1 - r^2)^{-1/p} \|F\|_p. \]
This is reduced to the case $p = 2$ in the standard way (see [2]). Let $F(z) = \sum c_nz^n$ belong to $H^2$. Then
\[ M_\infty^2(r, F) \leq \left( \sum_{0}^{\infty} |c_n|r^n \right)^2 \leq \sum_{0}^{\infty} |c_n|^2 \sum_{0}^{\infty} r^{2n} = \|F\|_2^2(1 - r^2)^{-1}, \]
and this concludes the proof.

PROOF OF THE THEOREM. It is easily shown that $f$ and $g$ may be supposed to be analytic in the closed disc. By Parseval’s formula
\[ h(r^2w) := (f * g)(r^2w) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{-it})g(re^{it}w) \, dt, \quad |w| = 1, \]
whence
\[ |h(r^2w)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{-it})||g(re^{it}w)| \, dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} |F(re^{it})| \, dt = M_1(r, F), \]
where
\[ F(z) = f(\overline{z})g(zw), \quad |z| \leq 1. \]
Since $f$ is analytic in the closed unit disc we may apply the lemma to obtain
\[ (1 - r^2)^{1-p}|h(r^2w)|^p \leq \|F\|_p^p \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{-it})|^p|g(e^{it}w)|^p \, dt. \]
Hence, by Minkowski’s inequality (in continuous form),
\[ (1 - r^2)^{1-p}M_s(r^2, |h|^p) \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{-it})|^p \, dt \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(e^{it}e^{i\theta})|^p \, d\theta \right\}^{1/s}, \]
where $s \geq 1$. By taking $s = q/p$ we get
\[ (1 - r^2)^{1-p}M_q^p(r^2, h) = (1 - r^2)^{1-p}M_s(r^2, |h|^p) \leq \|f\|_p^p\|g\|_q^p, \]
and this concludes the proof.
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