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INFINITE TENSOR PRODUCTS OF COMMUTATIVE SUBSPACE LATTICES

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ABSTRACT. Every infinite tensor product of commutative subspace lattices is unitarily equivalent to a certain lattice of projections on $L^2(X,\nu)$, where (X,ν) is an infinite product measure space. This representation reflects the structure of the individual component lattices in that the components of the tensor product correspond to the coordinates of the product space. This result generalizes the similar representation for finite tensor products. It is then used to show that an infinite tensor product of purely atomic commutative subspace lattices must be either purely atomic or noncompact, and in the latter case the algebra of operators under which the lattice is invariant has no compact operators.

If $\mathscr{L}$ is a commutative subspace lattice acting on a separable Hilbert space, then $\mathscr{L}$ can be represented as multiplication by certain characteristic functions on $L^2(X,\nu)$, where (X,ν) is a probability space. This is known as the Arveson representation [**A**, Theorem 1.3.1], and has proved to be very useful in the study of commutative subspace lattices. If $\mathscr{L}_i$, $i = 1,\dots,n$, are represented on $L^2(X_i,\nu_i)$, then it was shown in [**GHL**, Proposition 2.1] that $\bigotimes_{i=1}^n \mathscr{L}_i$ is represented in a natural way on $L^2(X,\nu)$, where $X = \prod_{i=1}^n X_i$ and $\nu = \prod_{i=1}^n \nu_i$. This result has also been useful (for example, see [**GHL**, **HLM**, and **K**]). In [**W**], the author extended this result to certain infinite tensor products, and then used it to show that "most" infinite tensor products of commutative subspace lattices are not compact in the strong operator topology. In Theorem 1 of this paper, we prove the corresponding representation for all infinite tensor products.

Theorem 2 is an application of this representation theorem which strengthens the second result in [**W**] mentioned above. One consequence is that an infinite tensor product of purely atomic lattices is either noncompact or purely atomic. In addition, if it is noncompact it actually satisfies a stronger condition which implies that every "piece" of the lattice is also noncompact, and that the reflexive algebra associated with the lattice has no compact operators.

It is hoped that this representation will make it easier to work with infinite tensor products, as the finite case did. We note that infinite tensor products have provided some important examples in the past. They give one of the essentially two types of commutative subspace lattices known which are not compact, and also one of the two types known which are not completely distributive [**W**]. They also provide examples of commutative lattices whose associated reflexive algebras are not hyperreflexive [**DP**].

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Every Hilbert space in this paper will be complex, infinite-dimensional, and separable. The set of all bounded operators on a Hilbert space $\mathscr{H}$ will be denoted $\mathscr{B}(\mathscr{H})$. All operators discussed will be bounded, and all projections will be selfadjoint. A *subspace lattice* on $\mathscr{H}$ is a lattice of projections which is closed in the strong operator topology and which contains 0 and I. Every subspace lattice is complete (closed under arbitrary intersections and closed linear spans). A subspace lattice is *commutative* if all the projections mutually commute. Every lattice in this paper will be a commutative subspace lattice (CSL). If $\mathscr{L}$ is a CSL and $A = P' - P$ for some $P, P' \in \mathscr{L}$, $P < P'$, then A is an *atom* of $\mathscr{L}$ if $QA = A$ or 0 for all $Q \in \mathscr{L}$. $\mathscr{L}$ is *purely atomic* if $I = \sum A_i$ where the A_i's are atoms of $\mathscr{L}$.

For each i, $i = 1, 2, \ldots$, let u_i be a unit vector in a separable Hilbert space $\mathscr{H}_i$, and let $u = (u_i)_{i=1}^{\infty}$. Let $\tilde{H}$ be the algebraic tensor product of the $\mathscr{H}_i$'s, and let $\mathscr{H}_0 \subset \tilde{\mathscr{H}}$ be the subspace of finite linear combinations of elements of the form $\bigotimes_{i=1}^{\infty} h_i$, with $h_i = u_i$ for almost every i (i.e., for all but a finite number). Define an inner product on $\mathscr{H}_0$ by setting $\langle \bigotimes h_i, \bigotimes k_i \rangle = \prod_{i=1}^{\infty} \langle h_i, k_i \rangle$ for such elements as above, and extending linearly. $\mathscr{H}_0$ is then a pre-Hilbert space, and the separable Hilbert space $\mathscr{H} = \bigotimes_{i=1}^{\infty_u} \mathscr{H}_i$ is defined to be the completion of $\mathscr{H}_0$ (see [**G** or **vN**] for details). If $\mathscr{R}_i$ is a von Neumann algebra acting on $\mathscr{H}_i$, then $\bigotimes_{i=1}^{\infty_u} \mathscr{R}_i$ is the von Neumann algebra on $\mathscr{H}$ generated by the operators of the form $\bigotimes_{i=1}^{\infty} T_i$ with $T_i \in \mathscr{R}_i$ and $T_i = I$ for almost every i. In particular, $\mathscr{B}(\mathscr{H}) = \bigotimes_{i=1}^{\infty_u} \mathscr{B}(\mathscr{H}_i)$. Now if $\mathscr{L}_i$ is a CSL on $\mathscr{H}_i$, we define $\bigotimes_{i=1}^{\infty_u} \mathscr{L}_i \subset \mathscr{B}(\mathscr{H})$ to be the subspace lattice on $\mathscr{H}$ generated by the projections of the form $\bigotimes_{i=1}^{\infty} P_i$ with $P_i \in \mathscr{L}_i$ and $P_i = I$ for almost every i. We remark also that $\mathscr{H}$ has a basis consisting of elements of the form $\bigotimes_{i=1}^{\infty} h_i$ with $h_i = u_i$ for almost every i. Thus, if $v_i \in \mathscr{H}_i'$ with $||v_i|| = 1$, $i = 1, 2, \ldots$, then $\bigotimes_{i=1}^{\infty_u} \mathscr{H}_i$ and $\bigotimes_{i=1}^{\infty_v} \mathscr{H}_i'$ are unitarily equivalent via a unitary operator constructed in the obvious way from unitaries U_i: $\mathscr{H}_i \to \mathscr{H}_i'$ with $U_i(u_i) = v_i$. Moreover, $\bigotimes_{i=1}^{\infty_u} \mathscr{L}_i \cong \bigotimes_{i=1}^{\infty_v} U_i \mathscr{L}_i U_i^{-1}$.

We now describe the Arveson representation from [**A**] in more detail. Let X be a compact metric space and μ a finite positive Borel measure on X. If g is a bounded Borel function on X, then we define $\lambda(g)$ to be the multiplication operator acting on $L^2(X, \mu)$ by $(\lambda(g)f)(x) = g(x)f(x)$. If E is a Borel set in X, then we use the notation $\lambda(E)$ instead of $\lambda(\chi_E)$. Now suppose $\leq$ is a partial order (a reflexive and transitive relation) in X whose graph is closed in $X \times X$. A Borel set $E \subseteq X$ is *increasing* if $x \in E$ and $x \leq y \Rightarrow y \in E$. Then $\mathscr{L}(X, \leq, \mu) = \{\lambda(E)\colon E$ is an increasing Borel set in $X\}$ is a subspace lattice. The Arveson representation states that if $\mathscr{L}$ is a CSL, then there is a partially ordered measure space $(X, \leq, \mu)$ as above such that $\mathscr{L}$ is unitarily equivalent to $\mathscr{L}(X, \leq, \mu)$. μ may be taken to be a probability measure.

Suppose $\mathscr{L}_i$, $i = 1, \ldots, n$, have Arveson representations $\mathscr{L}(X_i, \leq_i, \mu_i)$. Let $X = \prod_{i=1}^{n} X_i$ and $\mu = \prod_{i=1}^{n} \mu_i$. The product ordering $\leq$ in X is defined by $x \leq y \Leftrightarrow x_i \leq_i y_i$ for all i. The result in [**GHL**] referred to in the first paragraph states that $\bigotimes_{i=1}^{n} \mathscr{L}_i \cong \mathscr{L}(X, \leq, \mu)$. Now suppose that for each i, $i = 1, 2, \ldots$, X_i is a compact metric space, ν_i is a probability measure on X_i, and $\leq_i$ is a partial order in X_i with closed graph. Let $X = \prod_{i=1}^{\infty} x_i$, $\nu = \prod_{i=1}^{\infty} \nu_i$, and let $\leq$ be the product ordering in X. The lattice $\mathscr{L}(X, \leq, \nu)$ formed in this manner will be called an *infinite product lattice*. We will show that every infinite tensor product $\bigotimes_{i=1}^{\infty_u} \mathscr{L}_i$ is unitarily equivalent to an infinite product lattice $\mathscr{L}(X, \leq, \nu)$ (with, of course,

each $\mathscr{L}_i \cong \mathscr{L}(X_i, \leq_i, \nu_i)$). We refer to [**H**, §38] for the details on infinite product measure spaces. Also, recall that if $m_i > 0$, then $\prod_{i=1}^{\infty} m_i$ is positive and finite if and only if $\sum_{i=1}^{\infty} |1 - m_i|$ converges.

The following lemma, along with its proof, is actually the Arveson representation theorem with a few modifications that we need for our purposes.

LEMMA. *Suppose $\mathscr{L}$ is a CSL acting on $\mathscr{H}$, u is a unit vector in $\mathscr{H}$, and $\varepsilon > 0$. Then there is a compact metric space X, a closed partial order $\leq$ in X, a Borel measure μ on X with $1 \leq \mu(X) < 1 + \varepsilon$, and a unitary operator $U\colon L^2(X, \mu) \to \mathscr{H}$ such that $U^{-1}\mathscr{L}U = \mathscr{L}(X, \leq, \mu)$. Also, there is a closed and open set $F \subseteq X$ with $\mu(F) = 1$ such that $U(\chi_F) = u$.*

PROOF. Let $\mathscr{M}$ be a maximal abelian von Neumann algebra containing $\mathscr{L}$, and let M be the projection onto $[\mathscr{M}u]$, the closed linear span of $\{Tu : T \in \mathscr{M}\}$. Since $\mathscr{H}$ is separable, $\mathscr{L}$ and $\operatorname{lat}\mathscr{M} = \{\text{projections in } \mathscr{M}' = \mathscr{M}\}$ contain strongly dense sequences $\{P_1, P_2, \dots\}$ and $\{Q_1, Q_2, \dots\}$, respectively. We may assume that $\{Q_n\}$ contains I and M. Let $\mathscr{A}$ be the C^*-algebra generated by $\{P_n\} \cup \{Q_n\}$. Then $\mathscr{A}$ is a commutative C^*-algebra with identity which is norm separable and strongly dense in $\mathscr{M}$. Since $\mathscr{M}$ is maximal abelian and $\mathscr{H}$ is separable, $\mathscr{M}$ has a cyclic vector z. We can arrange that $z = u + v$ with $u \perp v$ and $\|v\| < \sqrt{\varepsilon}$. To see this, note that by Zorn's lemma there is a set of unit vectors $\{v_i : i \in I\}$ which contains u and which is maximal with respect to the property that $[\mathscr{M}v_i] \perp [\mathscr{M}v_j]$ for all $i \neq j$. By maximality, $\sum_i^{\oplus}[\mathscr{M}v_i] = \mathscr{H}$. Since $\mathscr{H}$ is separable, the index set I must be countable, so we may enumerate $\{v_i\}$ as a sequence $u = v_0, v_1, v_2, \dots$. Now let $v = \sum_{i=1}^{\infty} 2^{-i}\varepsilon^{1/4}v_i$ and let $z = u + v$. Clearly $u \perp v$ and $\|v\| < \sqrt{\varepsilon}$. If M_i denotes the projection onto $[\mathscr{M}v_i]$, then $M_i \in \mathscr{M}' = \mathscr{M}$, so $M_i z \in [\mathscr{M}z]$. Therefore, $v_i = 2^i\varepsilon^{-1/4}M_i z \in [\mathscr{M}z]$, and it follows that z is cyclic since $\mathscr{H} = \sum_i^{\oplus}[\mathscr{M}v_i]$.

Let X be the spectrum of $\mathscr{A}$. X is a compact Hausdorff space, and is second countable since $\mathscr{A} \cong C(X)$ is norm separable. Thus, X is metrizable. Let $\pi\colon C(X) \to \mathscr{A}$ be the inverse Gelfand map. Then there are sequences $\{E_n\}$, $\{F_n\}$ of closed and open sets in X such that $\pi(\chi_{E_n}) = P_n$ and $\pi(\chi_{F_n}) = Q_n$, $n = 1, 2, \dots$. Let F be the set such that $\pi(\chi_F) = M$. Define a partial order $\leq$ in X by $x \leq y$ if and only if $\chi_{E_n}(x) \leq \chi_{E_n}(y)$ for all $n \geq 1$. The graph $G = \{(y, x) : x \leq y\}$ is closed in $X \times X$ since its complement can be expressed as a union $\bigcup_{n=1}^{\infty}(X \backslash E_n) \times E_n$ of open rectangles.

By the Riesz-Markov theorem, there is a finite Borel measure μ on X such that $\int_X f\,d\mu = \langle \pi(f)z, z\rangle$ for $f \in C(X)$. Note that $\mu(X) = \|z\|^2 = \|u\|^2 + \|v\|^2 < 1 + \varepsilon$. Also, $\mu(F) = \int_X \chi_F\,d\mu = \langle Mz, z\rangle = \|u\|^2 = 1$. Define a linear map U_0 of $C(X)$ into $\mathscr{H}$ by $U_0 f = \pi(f)z$. Then $\|U_0 f\|^2 = \int_X |f|^2\,d\mu$, so U_0 extends uniquely to an isometry U of $L^2(X, \mu)$ onto $[\mathscr{A}z]$. Because $\mathscr{A}$ is strongly dense in $\mathscr{M}$ and $[\mathscr{M}z] = \mathscr{H}$, it follows that U is unitary. Note that $U(\chi_F) = \pi(\chi_F)z = Mz = u$.

Now if $g \in C(X)$, then

$$U\lambda(g)U^{-1}\pi(f)z = U\lambda(g)f = U(gf) = \pi(gf)z = \pi(g)\pi(f)z$$

for every $f \in C(X)$, so $U\lambda(g)U^{-1} = \pi(g)$. In particular, $U\lambda(E_n)U^{-1} = \pi(\chi_{E_n}) = P_n$. By [**A**, Theorem 1.2.2], $\mathscr{L}(X, \leq, \mu)$ is generated by $\{\lambda(E_n)\}$. It follows that $U\mathscr{L}(X, \leq, \mu)U^{-1} = \mathscr{L}$ since $\mathscr{L}$ is generated by $\{P_n\}$. $\square$

THEOREM 1. *Every infinite tensor product of CSL's is unitarily equivalent to an infinite product lattice.*

PROOF. For each $i = 1, 2, \ldots$, let $\mathscr{L}_i$ be a CSL acting on $\mathscr{H}_i$, and let u_i be a unit vector in $\mathscr{H}_i$. We must show that $\bigotimes_{i=1}^{\infty_u} \mathscr{L}_i$ is unitarily equivalent to an infinite product lattice. First use the lemma to get partially ordered measure spaces $(X_i, \leq_i, \mu_i)$ and unitaries U_i such that $1 \leq \mu_i(X_i) < 1 + 2^{-i}$ and $U_i^{-1}\mathscr{L}U_i = \mathscr{L}(X_i, \leq_i, \mu_i)$. Also, there are closed and open sets $F_i \subseteq X_i$ with $\mu_i(F_i) = 1$ such that $U_i^{-1}u_i = \chi_{F_i}$. Now let $m_i = \mu_i(X_i)$, define a measure ν_i on X_i by $\nu_i(E) = m_i^{-1}\mu_i(E)$, and define a unitary operator $V_i\colon L^2(X_i, \mu_i) \to L^2(X_i, \nu_i)$ by $V_i f = \sqrt{m_i} f$. Note that $V_i\lambda(E)V_i^{-1} = \lambda(E)$. Then if we let $\chi = (\chi_{F_i})_{i=1}^{\infty}$, $\varphi_i = \sqrt{m_i}\chi_{F_i}$, and $\varphi = (\varphi_i)_{i=1}^{\infty}$, it follows that

$$\bigotimes_{i=1}^{\infty_u} \mathscr{L}_i \cong \bigotimes_{i=1}^{\infty_\chi} \mathscr{L}(X_i, \leq_i, \mu_i) \cong \bigotimes_{i=1}^{\infty_\varphi} \mathscr{L}(X_i, \leq_i, \nu_i).$$

The remainder of the proof will show that $\bigotimes_{i=1}^{\infty_\varphi} \mathscr{L}(X_i, \leq_i, \nu_i) \cong \mathscr{L}(X, \leq, \nu)$, where $X = \prod_{i=1}^{\infty} X_i$, $\nu = \prod_{i=1}^{\infty} \nu_i$, and $\leq$ is the product ordering on X.

For $f_i \in L^2(X_i, \nu_i)$, $i = 1, \ldots, k$, define the following functions on X:

$$\mathscr{I}(f_1, \ldots, f_k) = \left(\prod_{i=1}^{k} f_i\right)\left(\prod_{i=k+1}^{\infty} 1\right),$$

$$\mathscr{F}(f_1, \ldots, f_k) = \left(\prod_{i=1}^{k} f_i\right)\left(\prod_{i=k+1}^{\infty} \chi_{F_i}\right),$$

$$\mathscr{G}(f_1, \ldots, f_k) = \left(\prod_{i=1}^{k} f_i\right)\left(\prod_{i=k+1}^{\infty} \varphi_i\right).$$

$\mathscr{F}(f_1, \ldots, f_k)$ and $\mathscr{G}(f_1, \ldots, f_k)$ are both ν-measurable functions since

$$\mathscr{F}(f_1, \ldots, f_k) = \lim_{n\to\infty} \mathscr{I}(f_1, \ldots, f_k, \chi_{F_{k+1}}, \ldots, \chi_{F_n}) \quad \text{and}$$

$$\mathscr{G}(f_1, \ldots, f_k) = \lim_{n\to\infty} \mathscr{F}(f_1, \ldots, f_k, \varphi_{k+1}, \ldots, \varphi_n).$$

It follows that $\mathscr{G}(f_1, \ldots, f_k) \in L^2(X, \nu)$ since $\prod_{i=1}^{\infty} m_i$ converges. Now recall that $\bigotimes_{i=1}^{\infty_\varphi} L^2(X_i, \nu_i)$ has a basis of elements of the form $\bigotimes_{i=1}^{\infty} f_i$ with $f_i = \varphi_i$ for almost every i. Thus, we can define an isometry $V\colon \bigotimes_{i=1}^{\infty_\varphi} L^2(X_i, \nu_i) \to L^2(X, \nu)$ by $V(\bigotimes f_i) = \mathscr{G}(f_1, \ldots, f_k)$ if $f_i = \varphi_i$ for $i > k$. For each n, let $\mathscr{B}_n$ be the σ-algebra generated by rectangles of the form $E_1 \times E_2 \times \cdots \times E_n \times X_{n+1} \times X_{n+2} \times \cdots$ (where each E_i is a Borel set in X_i). Then $\bigcup \mathscr{B}_n$ generates the σ-algebra of the product measure space (X, ν). By the increasing Martingale theorem [**D**, Theorem VII.4.1], it follows that $\bigcup L^2(X, \mathscr{B}_n, \nu|_{\mathscr{B}_n})$ is dense in $L^2(X, \nu)$. Thus, to show that V is unitary, it suffices to show that each $L^2(X, \mathscr{B}_n, \nu|_{\mathscr{B}_n}) \subseteq \mathrm{Range}(V)$.

Let $Y = \{(x_i)_{i=1}^{\infty} \in X\colon x_i \in F_i \text{ for almost every } i\} = \bigcup_k \{X_1 \times \cdots \times X_k \times F_{k+1} \times F_{k+2} \times \cdots\}$. Y is measurable since each $X_1 \times \cdots X_k \times F_{k+1} \times F_{k+2} \times \cdots$ is a countable intersection of measurable rectangles. Also, $\nu(X \backslash Y) = 0$ since

$$\nu(Y) = \lim_{k\to\infty} \nu(X_1 \times \cdots X_k \times F_{k+1} \times F_{k+2} \times \cdots) = \lim_{k\to\infty} \prod_{i=k+1}^{\infty} m_i^{-1} = 1 = \nu(X)$$

(because $\prod_{i=1}^{\infty} m_i^{-1}$ converges to a positive number). Now fix n and $f_i \in L^2(X_i, \nu_i)$, $i = 1, \ldots, n$, and let $f = \mathscr{I}(f_1, \ldots f_n)$. Let $g_m = \mathscr{G}(f_1, \ldots, f_n, 1_{n+1}, \ldots, 1_m)$ for $m > n$, where 1_i is the constant function 1 on X_i. Then

$$\|f - g_m\|^2 = \int_X |f - g_m|^2 \, d\nu = \int_Y |f - g_m|^2 \, d\nu = \int_X |\chi_Y f - g_m|^2 \, d\nu \to 0$$

by the dominated convergence theorem. Since each $g_m \in \mathrm{Range}(V)$, and $L^2(X, \mathscr{B}_n, \nu|_{\mathscr{B}_n})$ has a basis of functions of the form $\mathscr{I}(f_1, \ldots, f_n)$, it follows that $L^2(X, \mathscr{B}_n, \nu|_{\mathscr{B}_n}) \subseteq \mathrm{Range}(V)$. Therefore, V is unitary.

Finally, we need to show that $V(\bigotimes_{i=1}^{\infty\varphi} \mathscr{L}(X_i, \leq_i, \nu_i))V^{-1} = \mathscr{L}(X, \leq, \nu)$. For each i, let $P_i \in \mathscr{L}(X_i, \leq_i, \nu_i)$ and let E_i be a corresponding increasing Borel subset of X_i (i.e., $\lambda(E_i) = P_i$). Let $\widetilde{R} = P_1 \otimes \cdots \otimes P_n \otimes I \otimes I \otimes \cdots$ and $\widetilde{G} = E_1 \times \cdots \times E_n \times X_{n+1} \times X_{n+2} \times \cdots$. Then an easy calculation shows that $V\widetilde{R}V^{-1} = \lambda(\widetilde{G})$. Since $\lambda(\widetilde{G}) \in \mathscr{L}(X, \leq, \nu)$ and $\bigotimes_{i=1}^{\infty\varphi} \mathscr{L}(X_i, \leq_i, \nu_i)$ is generated by the projections of the form $P_1 \otimes \cdots \otimes P_n \otimes I \otimes I \otimes \cdots$, it follows that $V(\otimes_{i=1}^{\infty\varphi} \mathscr{L}(X_i, \leq_i, \nu_i))V^{-1} \subseteq \mathscr{L}(X, \leq, \nu)$.

Now for each i, let $\{E_{ij} \colon ! \leq j < \infty\}$ be a sequence of increasing Borel subsets of X_i such that $x_i \leq_i y_i \Leftrightarrow \chi_{E_{ij}}(x_i) \leq \chi_{E_{ij}}(y_i)$ for all j. Such sequences exist by [**A**, Proposition 1.1.2]. The sets of the form $\widetilde{D}_{ij} = X_1 \times \cdots \times X_{i-1} \times E_{ij} \times X_{i+1} \times X_{i+2} \times \cdots$ are increasing Borel subsets of X, and $x \leq y \Leftrightarrow \chi_{\widetilde{D}_{ij}}(x) \leq \chi_{\widetilde{D}_{ij}}(y)$ for all i, j. By [**A**, Theorem 1.2.2], $\mathscr{L}(X, \leq, \nu)$ is generated by $\{\lambda(\widetilde{D}_{ij})\}$. Let $P_{ij} = \lambda(E_{ij})$ and $\widetilde{R}_{ij} = I_1 \otimes \cdots \otimes I_{i-1} \otimes P_{ij} \otimes I_{i+1} \otimes I_{i+2} \otimes \cdots$. Then $V^{-1}\lambda(\widetilde{D}_{ij})V = \widetilde{R}_{ij} \in \bigotimes_{i=1}^{\infty\varphi} \mathscr{L}(X_i, \leq_i, \nu_i)$, so $V^{-1}(\mathscr{L}(X, \leq, \nu))V \subseteq \otimes_{i=1}^{\infty\varphi} \mathscr{L}(X_i, \leq_i, \nu_i)$. $\square$

REMARKS. (1) The lattice structure of each $\mathscr{L}_i$ is reflected in the representation since $\mathscr{L}_i \cong \mathscr{L}(X_i, \leq_i, \nu_i)$. However, the measure ν depends on the sequence $u = (u_i)$. To make full use of this theorem, it may be necessary to use the details of this dependence. If $P_i \in \mathscr{L}_i$, and E_i is a corresponding increasing set in X_i, then

$$\|P_i u_i\|^2 = m_i \int_{X_i} \chi_{E_i} \chi_{F_i} \, d\nu_i = m_i \cdot \nu_i(E_i \cap F_i) = \nu_i(E_i \cap F_i)/\nu_i(F_i).$$

(2) Let μ_i be as in the proof of the theorem, and let $\mu = \prod_{i=1}^{\infty} \mu_i$ (μ is defined by $\mu = (\prod_{i=1}^{\infty} m_i)\nu$). Then $\mathscr{L}(X, \leq, \nu) \cong \mathscr{L}(X, \leq, \mu)$ via the obvious unitary, so $\bigotimes_{i=1}^{\infty u} \mathscr{L}_i \cong \mathscr{L}(X, \leq, \mu)$. In this case, the u_i's correspond to characteristic functions χ_{F_i} and the above calculation becomes $\|P_i u_i\|^2 = \mu_i(E_i \cap F_i)$. The disadvantage is that the product space (X, μ) does not have measure 1.

(3) If $\mathscr{L}$ is a CSL on $\mathscr{H}$ and M is a projection in $\mathscr{L}'$, then $\mathscr{L}\, M$, acting on $M\mathscr{H}$, is also a CSL. We denote this lattice by $\mathscr{L}^M$ and call it an *induced lattice of* $\mathscr{L}$. Now suppose, for each i, $\mathscr{L}_i$ is a CSL on $\mathscr{H}_i$, u_i is a unit vector in $\mathscr{H}_i$, $\mathscr{M}_i$ is a maximal abelian von Neumann algebra containing $\mathscr{L}_i$, and M_i is the projection onto $[\mathscr{M}_i u_i]$. Then it follows from the proof of the lemma that $(\mathscr{L}_i)^{\mathscr{M}_i} \cong \mathscr{L}(F_i, \leq_i, \mu_i|_{F_i})$, where F_i and μ_i are as in the proof of the theorem. The proof of the theorem (much simpler in this case) then shows that $\bigotimes_{i=1}^{\infty u} (\mathscr{L}_i)^{M_i} \cong \mathscr{L}(\widetilde{F}, \leq, \mu|_{\widetilde{F}})$, where $\widetilde{F} = \prod_{i=1}^{\infty} F_i$. Note that $\mathscr{L}(\widetilde{F}, \leq, \mu|_{\widetilde{F}})$ is an induced lattice of $\mathscr{L}(X, \leq, \mu)$. It was this special case of the theorem that was used in [**W**].

(4) Another special case arises when, for almost every i, u_i is cyclic for some maximal abelian algebra $\mathscr{M}_i$ containing $\mathscr{L}_i$. In this case, if we let $I = \{i \colon u_i$ is

cyclic}, then $F_i = X_i$ and $\mu(X_i) = 1$ for all $i \in I$. For $i \notin I$, we use any *unit* cyclic vector z_i to get a representation $\mathscr{L}_i \cong \mathscr{L}(X_i, \leq_i, \mu_i)$ as in [**A**, Theorem 1.3.1]. The proof of the theorem (again, a much simpler case) then implies that $\bigotimes_{i=1}^{\infty u} \mathscr{L}_i \cong \mathscr{L}(X, \leq, \mu)$. This special case was also proved in [**W**].

We will now apply Theorem 1 to obtain a sufficient condition for an infinite tensor product $\mathscr{L}$ of CSL's to be noncompact in the strong operator topology. In addition, it will be shown that in this case $\mathscr{L}$ actually satisfies a "super" noncompactness property which implies that every induced lattice of $\mathscr{L}$ is also noncompact, and that the reflexive algebra $\operatorname{alg}\mathscr{L} = \{T \in \mathscr{B}(\mathscr{H}): TP\mathscr{H} \subseteq P\mathscr{H} \text{ for all } P \in \mathscr{L}\}$ contains no compact operators. This is most interesting in the case in which each component lattice of the tensor product is purely atomic, because then the condition is also necessary. Specifically, an infinite tensor product $\mathscr{L}$ of purely atomic CSL's either has the "super" noncompactness property or else is purely atomic. We note that purely atomic lattices must be compact [**W**, Corollary 2.4], but the converse is false. Also, if $\mathscr{L}$ is purely atomic, then the compact operators in $\operatorname{alg}\mathscr{L}$ are ultrastrongly dense in $\operatorname{alg}\mathscr{L}$. Finally, we remark that the conclusions of Theorem 2 are almost the same for orthocomplemented CSL's [**W**, Corollary 3.5].

In the remainder of this paper, (ω_i) will always denote a sequence of 0's and 1's, and E^c will denote the complement of a set E. The proof of Theorem 2 uses the following.

PROPOSITION. *Let $\mathscr{L}(X, \leq, \nu)$ be an infinite tensor product lattice, and suppose there is a sequence of increasing Borel sets $\{E_i \subseteq X_i\colon 1 \leq i < \infty\}$, with $0 < \nu_i(E_i) < 1$ for all i, such that $\sum_{i=1}^{\infty} |\omega_i - \nu_i(E_i)|$ does not converge for every sequence (ω_i) of 0's and 1's. Then $\mathscr{L}(X, \leq, \nu)$ is noncompact, every induced lattice of $\mathscr{L}(X, \leq, \nu)$ is noncompact, and $\operatorname{alg}\mathscr{L}(X, \leq, \nu)$ has no compact operators. In particular, the conclusion holds if there is a subsequence $\{E_{i_j}\}$ such that $\{\nu_{i_j}(E_{i_j})\}$ is bounded away from 0 and 1.*

PROOF. The noncompactness conclusion is just a restatement of Theorem 3.9(b) of [**W**], but a slightly modified argument and the addition of Theorem 1 will provide the stronger results. First, let $\widetilde{E}_i = X_1 \times \cdots \times X_{i-1} \times E_i \times X_{i+1} \times \cdots$, let $\mathscr{N}$ be the sublattice generated by $\{\lambda(\widetilde{E}_i)\}$, and suppose $A = \lambda(\widetilde{A})$ is an atom of $\mathscr{N}$. Then for each i, either $\widetilde{E}_i \cap \widetilde{A} = \widetilde{A}$ almost everywhere or $\widetilde{E}_i^c \cap \widetilde{A} = \widetilde{A}$ a.e. It follows that $\widetilde{A}$ is a set of the form $D_1 \times D_2 \times \cdots$, where $D_i =$ either E_i or E_i^c a.e. But $\nu(\widetilde{A}) = \prod_{i=1}^{\infty} \nu_i(D_i) = \prod_{i=1}^{\infty} |\omega_i - \nu_i(E_i)|$ for some (ω_i), and this product is 0 since by hypothesis $\sum_{i=1}^{\infty} |(1-\omega_i) - \nu_i(E_i)| = \sum_{i=1}^{\infty} 1 - |\omega_i - \nu_i(E_i)|$ does not converge. Therefore, $\mathscr{N}$ is nonatomic.

Now partition $\mathbb{N}$ into an infinite number of infinite subsets $S_1, S_2, \ldots$, such that for each j, $\sum_{i \in S_j} |\omega_i - \nu_i(E_i)|$ does not converge for every $(\omega_i)_{i \in S_j}$. Let $\mathscr{N}_j$ be the sublattice of $\mathscr{L}(X, \leq, \nu)$ generated by $\{\lambda(\widetilde{E}_i)\colon i \in S_j\}$. By the same argument as above, each $\mathscr{N}_j$ is nonatomic, and therefore there is for each j an increasing set $G_j \subseteq X$ with $\lambda(G_j) \in \mathscr{N}_j$ and $\nu(G_j) = \frac{1}{2}$. Then the sequence $\{\lambda(G_j)\colon 1 \leq j < \infty\}$ has the property that $\bigvee_k \lambda(G_{j_k}) = I$ and $\bigwedge_k \lambda(G_{j_k}) = 0$ for every subsequence $\{\lambda(G_{j_k})\}$, since $\prod_{k=1}^{\infty} \nu(G_{j_k}) = 0 = \prod_{k=1}^{\infty} \nu(G_{j_k}^c)$. It follows that $\mathscr{L}(X, \leq, \nu)$ is noncompact by [**W**, Theorem 3.6] and that $\operatorname{alg}\mathscr{L}(X, \leq, \nu)$ has no

compact operators by [**F**]. Also, if M is a projection in $\mathscr{L}'$, then $\bigvee_k \lambda(G_{j_k})M = M = I_{M\mathscr{H}}$ and $\bigwedge_k \lambda(G_{j_k})M = 0$ for every subsequence $\{\lambda(G_{j_k})\}$, so the induced lattice $\mathscr{L}(X, \leq, \nu)^M$ is also noncompact. □

THEOREM 2. *Suppose that $\mathscr{L}_i$ is a (nontrivial) commutative subspace lattice on $\mathscr{H}_i$ and u_i is a unit vector in $\mathscr{H}_i$, $i = 1, 2, \ldots$.*

(a) *If there is a sequence $\{P_i \in \mathscr{L}_i : 0 < P_i < I\}$ such that $\sum_{i=1}^{\infty} |\omega_i - \|P_i u_i\|^2|$ does not converge for every sequence (ω_i) of 0's and 1's, then $\mathscr{L} = \bigotimes_{i=1}^{\infty u} \mathscr{L}_i$ is noncompact, every induced lattice of $\mathscr{L}$ is also noncompact, and* alg $\mathscr{L}$ *contains no compact operators. In particular, the conclusion holds if there is a subsequence $\{P_{i_j}\}$ such that $\{\|P_{i_j} u_{i_j}\|\}$ is bounded away from* 0 *and* 1.

(b) *If each $\mathscr{L}_i$ is purely atomic and there is no such sequence as in part* (a), *then $\mathscr{L} = \bigotimes_{i=1}^{\infty u} \mathscr{L}_i$ is purely atomic and the subalgebra generated by the rank-one operators in* alg $\mathscr{L}$ *is ultrastrongly dense in* alg $\mathscr{L}$.

PROOF. (a) By Theorem 1, $\mathscr{L}$ is unitarily equivalent to an infinite product lattice $\mathscr{L}(X, \leq, \nu)$, with $X = \prod_{i=1}^{\infty} X_i$, $\nu = \prod_{i=1}^{\infty} \nu_i$, and each $\mathscr{L}_i \cong \mathscr{L}(X_i, \leq_i, \nu_i)$. Let F_i and μ_i be as in the proof of Theorem 1, and let $E_i \subseteq X_i$ be an increasing Borel set corresponding to P_i (thus, $0 < \nu_i(E_i) < 1$). Then

$$\begin{aligned}
\sum |\omega_i - \nu_i(E_i)| &= \sum \left| \omega_i - \frac{\mu_i(E_i)}{\mu_i(X_i)} \right| \\
&= \frac{1}{\mu_i(x_i)} \sum |\mu_i(X_i)\omega_i - \mu_i(E_i)| \\
&\geq \frac{2}{3} \sum |\mu_i(X_i)\omega_i - \mu_i(E_i)| \quad \text{since } \mu_i(X_i) < 1 + 2^{-i} \leq \frac{3}{2} \\
&\geq \frac{2}{3} \sum |\mu_i(F_i)\omega_i - \mu_i(E_i \cap F_i)| \\
&= \frac{2}{3} \sum |\omega_i - \mu_i(E_i \cap F_i)| \\
&= \frac{2}{3} \sum |\omega_i - \|P_i u_i\|^2| \quad \text{by Remark (2)},
\end{aligned}$$

so $\sum |\omega_i - \nu_i(E_i)|$ does not converge for every (ω_i). The result now follows by the preceding proposition.

(b) As in part (a), we apply Theorem 1 and consider the infinite product lattice $\mathscr{L}(X, \leq, \nu) \cong \mathscr{L}$. Let $\mathscr{B}(X)$ and $\mathscr{B}(X_i)$ denote the nontrivial (neither null nor conull) increasing Borel sets in X and X_i, respectively. By hypothesis, for every sequence of projections $\{P_i \in \mathscr{L}_i : 0 < P_i < I\}$, there is a sequence (ω_i) of 0's and 1's such that $\sum |\omega_i - \|P_i u_i\|^2|$ converges. Then by calculations similar to those in part (a), it follows that for every sequence of sets $\{E_i : E_i \in \mathscr{B}(X_i)\}$, there is a sequence (ω_i) such that $\sum |\omega_i - \nu_i(E_i)|$ converges. This implies that for every such sequence $\{E_i\}$, the sequence $\{\nu_i(E_i)\}$ has at most the two limit points 0 and 1, and therefore there is some I with the property that if $i \geq I$ and $E \in \mathscr{B}(X_i)$, then either $\nu_i(E) \geq \frac{3}{4}$ or $\nu_i(E) \leq \frac{1}{4}$. Since it is enough to prove that $\bigotimes_{i=I}^{\infty} \mathscr{L}_i$ is purely atomic (because the finite tensor product of purely atomic lattices is purely atomic), we can now restrict our attention to $\prod_{i=I}^{\infty} X_i$. Equivalently, we may assume $I = 1$.

For each i, let $r_i = \inf\{\nu_i(E) : E \in \mathscr{B}(X_i) \text{ and } \nu_i(E) \geq \frac{3}{4}\}$. Then for each i, there is a sequence $\{G_j\} \subseteq \mathscr{B}(X_i)$ such that $\nu_i(G_j) \to r_i$. Since $\mathscr{L}(X_i, \leq_i, \nu_i)$

is purely atomic, it is compact [**W**, Corollary 2.4], so there is a set $R_i \in \mathscr{B}(X_i)$ and a subsequence $\{G_{j_k}\}$ such that $\lambda(G_{j_k}) \to \lambda(R_i)$ strongly. This implies that $\nu_i(G_{j_k} \Delta R_i) \to 0$ [**W**, §3], so $\nu_i(G_{j_k}) \to \nu_i(R_i)$ and $\nu_i(R_i) = r_i$. Now let $s_i = \sup\{\nu_i(E)\colon E \in \mathscr{B}(X_i),\ \nu_i(E) \le \frac{1}{4}, \text{ and } \nu_i(E \backslash R_i) = 0\}$. Then by a similar argument, there is a set $S_i \in \mathscr{B}(X_i)$ with $\nu_i(S_i \backslash R_i) = 0$ and $\nu_i(S_i) = s_i$. Let $A_i = R_i \backslash S_i$ and let E be any set in $\mathscr{B}(X_i)$. If $E \cap A_i \neq A_i$ a.e., then $\nu_i(E \cap A_i) \le \frac{1}{4}$, so $\nu_i(S_i \cup (E \cap A_i)) \le \frac{1}{2}$ and thus $\nu_i(S_i \cup (E \cap A_i)) \le s_i$. This implies that $E \cap A_i \subseteq S_i$ a.e., so $E \cap A_i = \varnothing$ a.e. Therefore, $E \cap A_i = \varnothing$ or A_i a.e. for any $E \in \mathscr{B}(X_i)$, and it follows that $\lambda(A_i)$ is an atom of $\mathscr{L}(X_i, \le_i, \nu_i)$.

Let $\widetilde{A}_k = A_k \times A_{k+1} \times \cdots \subseteq \prod_{i=k}^{\infty} X_i$, $k = 1, 2, \ldots$. Now $\prod_{i=1}^{\infty} \nu_i(R_i) > 0$ since $\sum_{i=1}^{\infty}(1 - \nu_i(R_i))$ converges (by hypothesis, $\sum |\omega_i - \nu_i(R_i)|$ converges for some (ω_i), and every choice of (ω_i) other than $\omega_i = 1$ for all i results in a larger sum). Similarly, $\prod_{i=1}^{\infty} \nu_i(S_i^c) > 0$ since $\sum \nu_i(S_i)$ converges. It follows that

$$\sum (1 - (\nu_i(R_i) - \nu_i(S_i))) = \sum (1 - \nu_i(A_i))$$

converges, and therefore $\nu((X_1 \times \cdots \times X_{k-1}) \times \widetilde{A}_k) = \prod_{i=k}^{\infty} \nu_i(A_i) > 0$. Thus, if $\lambda(C_i)$, $i = 1, \ldots, k-1$, are atoms of $\mathscr{L}(X_i, \le_i, \nu_i)$, then $\lambda((C_1 \times \cdots \times C_{k-1}) \times \widetilde{A}_k)$ is an atom of $\mathscr{L}(X, \le, \nu)$.

Finally, let $\{\lambda(C_{ij})\colon 1 \le j < \infty\}$ denote all the atoms of $\mathscr{L}(X_i, \le_i, \nu_i)$, with $C_{i1} = A_i$ a.e., and let $\widetilde{A}_k$ be as above. To show that $\mathscr{L}(X, \le, \nu)$ is purely atomic, it suffices to show that $\sum \lambda((C_{1j_1} \times \cdots \times C_{k-1,j_{k-1}}) \times \widetilde{A}_k) = I$, where the sum is taken over all $k \ge 1$ and all $(k-1)$-tuples $(j_1, \ldots, j_{k-1})$ such that $j_{k-1} > 1$. This is equivalent to showing that $\nu(\bigcup((C_{1j_1} \times \cdots \times C_{k-1,j_{k-1}})) \times \widetilde{A}_k) = 1$, and if we let $\widetilde{B}_0 = \widetilde{A}_1$ and $\widetilde{B}_k = \{(x_i)\colon x_i \in A_i \text{ for all } i > k \text{ and } x_k \notin A_k\}$ then it is also equivalent to $\nu(\bigcup_{k \ge 0} \widetilde{B}_k) = 1$. But if we let $p_i = \nu_i(A_i)$, then

$$\begin{aligned}
\nu\left(\bigcup \widetilde{B}_k\right) &= \sum_{k \ge 0} \nu(\widetilde{B}_k) \\
&= \prod_{i=1}^{\infty} p_i + (1 - p_1) \prod_{i=2}^{\infty} p_i + (1 - p_2) \prod_{i=3}^{\infty} p_i + \cdots \\
&= \lim_{k \to \infty} \prod_{i=k}^{\infty} p_i = 1 \quad \text{since } \prod_{i=1}^{\infty} p_i > 0.
\end{aligned}$$

The final statement of the theorem concerning the rank-one subalgebra follows from [**LL**, Theorem 3] and the fact that purely atomic CSL's are completely distributive. □

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