

LOCAL UNCERTAINTY INEQUALITIES FOR COMPACT GROUPS

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ABSTRACT. Conditions are established on $\alpha, \beta \in \mathbf{R}$ for there to exist a constant $K = K(\alpha, \beta)$ such that

$$\sum_{\gamma \in E} d(\gamma) \operatorname{tr}(\hat{f}(\gamma)^* \hat{f}(\gamma)) \leq K \left(\sum_{\gamma \in E} d(\gamma)^2 \right)^\alpha \|w^\beta f\|_2$$

for all $f \in L^1(G)$ and $E \subseteq \hat{G}$ where G is a compact metric group, \hat{G} is its dual, \hat{f} is the Fourier transform of f and $w: G \rightarrow \mathbf{R}^+$ is the function taking $x \in G$ to the area of the ball in G with centre e and x on its boundary. This is followed by a partial analogy for compact riemannian manifolds.

1. Introduction. The following is a special case of a result in [6] for multiple Fourier series: given $\alpha, \beta \in \mathbf{R}$ and $k \in \mathbf{Z}^+ = \{1, 2, \dots\}$, there exists a constant K such that

$$(1.1) \quad \left(\sum_{n \in E} |\hat{f}(n)|^2 \right)^{1/2} \leq K |E|^\alpha \| |x|^{k\beta} f \|_2$$

for all $f \in L^1(\mathbf{T}^k)$ and all finite $E \subseteq \mathbf{Z}^k$ if and only if α, β satisfy

$$(1.2) \quad \beta < 1/2, \quad \alpha \geq 0 \quad \text{and} \quad \alpha \geq \beta.$$

($|E|$ denotes the cardinality of E and the function $|x|$ is defined on \mathbf{T}^k by identifying this group with $(-\frac{1}{2}, \frac{1}{2}]^k$.) This is a local uncertainty inequality in the sense that concentration of f limits the localization of \hat{f} on any given set. The main result below, Theorem 2.4, is a direct analogue valid for all compact metrizable groups. We then give a somewhat less complete version for compact analytic manifolds. Local uncertainty inequalities for certain noncompact Lie groups are given in [7] and for \mathbf{R}^d in [1, 4, 5].

2. Compact metric groups. Throughout this section G will be a compact nonfinite metric group equipped with normalized Haar measure $d\mu$ and \hat{G} will be its unitary dual, that is, \hat{G} is a maximal set of pairwise inequivalent unitary irreducible continuous representations of G . Denote by \mathcal{H}_γ the (finite-dimensional) Hilbert space on which $\gamma \in \hat{G}$ acts. As usual, the Fourier series of $f \in L^1(G)$ is written as

$$f \sim \sum_{\gamma \in \hat{G}} d(\gamma) \operatorname{tr}(\hat{f}(\gamma) \gamma(\cdot)),$$

where $d(\gamma)$ is the dimension of \mathcal{H}_γ and $\hat{f}(\gamma) = \int_G f(x) \gamma(x^{-1}) d\mu(x)$. Our first concern is to introduce a function which plays the role of $|x|$ when $G = \mathbf{T}^k$.

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Let $d(\cdot, \cdot)$ be a metric on G which describes its topology. Without loss of generality assume that the metric is normalized so that $\sup\{d(e, x) : x \in G\} = 1$. Since G is compact, this supremum is actually attained. Denote $[0, 1]$ by I and define a nondecreasing measurable function $A : I \rightarrow I$ by

$$A(r) = \mu(B_r) \quad \text{where } B_r = \{x \in G : d(x, e) \leq r\}.$$

Since G is nonfinite and compact, $A(0) = 0$. Also the fact that if $r_n \searrow r$ as $n \rightarrow \infty$ for some sequence (r_n) , then $B_r = \bigcap_n B_{r_n}$, shows that A is continuous on the right.

Define a continuous map $\phi : G \rightarrow I$ by $\phi(x) = d(x, e)$. Also let B'_r denote the complement of B_r in G .

2.1 LEMMA. For any $\varepsilon > 0$, $\mu\{x \in G : A(\phi(x)) < \varepsilon\} \leq \varepsilon$.

REMARK. A is right continuous, ϕ is continuous and so $A \circ \phi$ is measurable. Hence the set in Lemma 2.1 is measurable.

PROOF OF 2.1. Given $\varepsilon > 0$, let $Y = \{x \in G : A(\phi(x)) < \varepsilon\}$. Since always $y \in B_{\phi(y)}$, $Y \subseteq \bigcup\{B_{\phi(x)} : x \in Y\}$. On the other hand, suppose $y \in \bigcup\{B_{\phi(x)} : x \in Y\}$, that is, $y \in B_{\phi(x)}$ for some $x \in Y$. Hence $\phi(y) \leq \phi(x)$ and so $A(\phi(y)) \leq A(\phi(x)) < \varepsilon$, with the conclusion that $y \in Y$. This has established the fact that

$$Y = \bigcup\{B_{\phi(x)} : x \in Y\},$$

from which the conclusion in Lemma 2.1 is a straightforward consequence.

2.2 LEMMA. The function $w = A \circ \phi : G \rightarrow I$ is measurable and satisfies

$$\int_{B_r} w^{-\theta} d\mu \leq \frac{A(r)^{1-\theta}}{1-\theta} \quad \text{for } 0 \leq \theta < 1$$

and

$$\|w^{-\theta} 1_{B'_r}\|_{\infty} \leq A(r)^{-\theta} \quad \text{for } \theta \geq 0$$

for each $r > 0$.

Further, for $\theta \leq 0$, $w^{-\theta}$ is continuous and hence bounded since G is compact. Consequently

$$\int_G w^{-\theta} d\mu < \infty \quad \text{for } \theta \leq 0.$$

PROOF. We just give a sketch of the proof of the first inequality. Define $G_t = \{x \in G : w(x)^{-1} > t\}$ for $t \geq 0$. By the change of variable formula [3, (21.72)],

$$(2.1) \quad \int_{B_r} w^{-\theta} d\mu = \int_0^{\infty} \theta t^{\theta-1} \mu(G_t \cap B_r) dt.$$

First consider the integral $I_1 = \int_0^{1/A(r)} \theta t^{\theta-1} \mu(G_t \cap B_r) dt$. Since $\mu(G_t \cap B_r) \leq \mu(B_r) = A(r)$, we have

$$(2.2) \quad I_1 \leq A(r) \int_0^{1/A(r)} \theta t^{\theta-1} dt = A(r)^{1-\theta}.$$

Now consider $I_2 = \int_{1/A(r)}^{\infty} \theta t^{\theta-1} \mu(G_t \cap B_r) dt$. Whenever $t > 1/A(r)$, $G_t \subseteq B_r$. (Let $x \in G_t$; then $A(\phi(x)) < t^{-1} < A(r)$ and so $\phi(x) \leq r$, that is, $x \in B_r$.) Hence

$$I_2 = \int_{1/A(r)}^{\infty} \theta t^{\theta-1} \mu(G_t) dt.$$

Lemma 2.1 shows that $\mu(G_t) \leq 1/t$ and so

$$(2.3) \quad I_2 \leq \int_{1/A(r)}^\infty \theta t^{\theta-1} t^{-1} dt = \frac{A(r)^{1-\theta\theta}}{1-\theta}.$$

Combining (2.2) and (2.3) and substituting in (2.1) gives the required inequality.

REMARK. In the above, notice that $A(r) > 0$ for $r > 0$. This is because B_r is a neighbourhood of e for each $r > 0$ and so has positive Haar measure.

2.3 ASSUMPTION. To obtain a more complete analogy with the result (1.1) for \mathbf{T}^k we will need G and its metric to satisfy the following: there exists $\lambda > 0$ such that for all $s \in [0, 1]$ there exists $r \in [0, 1]$ with

$$(2.4) \quad s \leq A(r) \leq \lambda s.$$

A wide class of groups, including the connected compact Lie groups, can be equipped with compatible metrics so that this condition is satisfied.

Whenever $E \subseteq \hat{G}$, define $|E|_2 = (\sum_{\gamma \in E} d(\gamma)^2)^{1/2}$.

2.4 THEOREM. Let G be a compact metric group and suppose $\alpha, \beta \in \mathbf{R}$. Consider the following inequality: there exists a constant $K = K(\alpha, \beta)$ such that

$$(2.5) \quad \left(\sum_{\gamma \in E} d(\gamma) \text{tr}(\hat{f}(\gamma)\hat{f}(\gamma)^*) \right)^{1/2} \leq K |E|_2^{2\alpha} \|w^\beta f\|_2$$

for all finite $E \subseteq \hat{G}$ and all $f \in L^1(G)$.

(i) The inequality is valid for $\{(\alpha, \beta): \alpha \geq 0, \beta < 1/2 \text{ and } \beta < \alpha\}$ and $\{(\alpha, \beta): \alpha = 0, \beta \leq 0\}$.

(ii) If G also satisfies Assumption 2.3 the inequality continues to hold when $0 < \alpha = \beta < 1/2$.

2.5 REMARK. When $G = \mathbf{T}^k$ the function $w = A \circ \phi$ can be chosen to equal $|x|^k$. Furthermore, in this case $|E|_2$ reduces to $|E|$.

PROOF (OF THEOREM 2.4). We first introduce spaces which are nonabelian analogues of $l^p(\mathbf{Z})$. Full details are available in Hewitt and Ross [2]. Let \mathfrak{E} be the set of functions ψ on \hat{G} with $\psi(\gamma) \in \mathcal{B}(\mathcal{X}_\gamma)$ for $\gamma \in \hat{G}$, where $\mathcal{B}(\mathcal{X}_\gamma)$ is the space of bounded linear operators on \mathcal{X}_γ . For $1 \leq p \leq \infty$, let \mathfrak{E}_p be the normed subspace of \mathfrak{E} as in [2, (28.24)]: denote the corresponding norm by $\|\cdot\|_p$. In particular, $\|\psi\|_2 = (\sum_{\gamma \in \hat{G}} d(\gamma) \text{tr}(\psi(\gamma)^* \psi(\gamma)))^{1/2}$ and $\|\psi\|_\infty = \sup_{\gamma \in \hat{G}} \|\psi(\gamma)\|$, where $\|\psi(\gamma)\|$ is the operator norm of $\psi(\gamma)$. Let E be a finite subset of \hat{G} .

(i) Throughout the proof of part (i) we assume that $\alpha, \beta \in \mathbf{R}$ satisfy $\alpha \geq 0$ and $\beta < 1/2$. Define $\psi_E \in \mathfrak{E}$ by $\psi_E(\gamma) = I_{d(\gamma)}$, the identity operator in $\mathcal{B}(\mathcal{X}_\gamma)$, when $\gamma \in E$ and 0 otherwise. For $p \in [1, \infty]$ define p' and $p^\#$ by $p' = p(p-1)^{-1}$ and $p^\# = 2p(p-2)^{-1}$.

Given $f \in L^1$, the following sequence of inequalities follows from (28.33) and (31.22) of [2] and Hölder’s inequality:

$$\begin{aligned} \left(\sum_{\gamma \in E} d(\gamma) \operatorname{tr}(\hat{f}(\gamma) \hat{f}(\gamma)^*) \right)^{1/2} &= \|\psi_E \hat{f}\|_2 \\ &\leq \|\psi_E\|_{p^\#} \|\hat{f}\|_p \quad (\text{where } 2 \leq p \leq \infty) \\ &\leq \|\psi_E\|_{p^\#} \|f\|_{p'} \\ &\leq \left(\sum_{\gamma \in E} d(\gamma)^2 \right)^{1/p^\#} \|(A \circ \phi)^{-\beta}\|_{p^\#} \|(A \circ \phi)^\beta f\|_2. \end{aligned}$$

Let $\alpha = 1/p^\#$. By Lemma 2.2, $\|(A \circ \phi)^{-\beta}\|_{p^\#}$ is finite when $\beta p^\# < 1$ (that is, when $\beta < \alpha$) and $\alpha = 1/p^\# > 0$, and when $\beta \leq 0$ and $\alpha = 0$.

In the preceding argument we required $2 \leq p \leq \infty$ which implies $0 \leq \alpha \leq 1/2$. Hence $\beta < 1/2$ is also required. Thus for the pairs $\{(\alpha, \beta): 0 \leq \alpha \leq 1/2, \beta < 1/2, \beta < \alpha\}$ and $\{(0, \beta): \beta \leq 0\}$ we have the required inequality with the constant $K = \|(A \circ \phi)^{-\beta}\|_{1/\alpha}$. Since $|E|_2 \geq 1$ for nonempty E , $|E|_2^{2\alpha} \leq |E|_2^{2\alpha'}$ whenever $\alpha \leq \alpha'$. This completes part (i) because the validity of the inequality (2.5) for a pair (α, β) implies its validity for all pairs (α', β) , with $\alpha' \geq \alpha$, with the same constant.

(ii) Up until the last step, the proof of part (ii) follows that of Theorem 1’ of [4] or Theorem 1.1 of [7]. We then invoke (2.4). Given $r \in (0, 1)$, let $f_1 = f \mathbf{1}_{B_r}$ and $f_2 = f - f_1$. Then

$$\left(\sum_{\gamma \in E} d(\gamma) \operatorname{tr}(\hat{f}(\gamma)^* \hat{f}(\gamma)) \right)^{1/2} \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \left(\sum_{\gamma \in E} d(\gamma) \operatorname{tr}(\hat{f}_1(\gamma)^* \hat{f}_1(\gamma)) \right)^{1/2} \\ &\leq |E|_2 \|\hat{f}_1\|_\infty \leq |E|_2 \|f_1\|_1 \leq |E|_2 \|w^{-\beta}\|_2 \|w^\beta f_1\|_2 \\ &\leq |E|_2 \frac{A(r)^{-\beta+1/2}}{(1-2\beta)^{1/2}} \|w^\beta f_1\|_2 \end{aligned}$$

and

$$\begin{aligned} I_2 &= \left(\sum_{\gamma \in E} d(\gamma) \operatorname{tr}(\hat{f}_2(\gamma)^* \hat{f}_2(\gamma)) \right)^{1/2} \\ &\leq \|f_2\|_2 \leq \|w^{-\beta} \mathbf{1}_{B_r^c}\|_\infty \|w^\beta f_2\|_2 \leq A(r)^{-\beta} \|w^\beta f_2\|_2. \end{aligned}$$

(In both cases the final inequality follows from Lemma 2.2.) Hence

$$(2.6) \quad \left(\sum_{\gamma \in E} d(\gamma) \operatorname{tr}(\hat{f}(\gamma)^* \hat{f}(\gamma)) \right)^{1/2} \leq A(r)^{-\beta} \left(\frac{A(r)^{1/2} |E|_2}{(1-2\beta)^{1/2}} + 1 \right) \|w^\beta f\|_2,$$

using the fact that $\|w^\beta f_1\|_2, \|w^\beta f_2\|_2 \leq \|w^\beta f\|_2$.

The proof is completed by applying inequality (2.4) (that is, Assumption 2.3) with $s = |E_2|^{-2}$. If E is nonempty (which we can of course assume), then $|E|_2 \geq 1$ and so $s = |E_2|^{-2} \leq 1$. Thus by inequality (2.4), there exists r such that $A(r) \leq \lambda|E|_2^{-2}$ and $A(r) \geq |E|_2^{-2}$. Thus $A(r)^{1/2} \leq \lambda^{1/2}|E|_2^{-1}$ and $A(r)^{-\beta} \leq |E|_2^{2\beta}$ which, upon substitution into (2.6), gives

$$\left(\sum_{\gamma \in E} d(\gamma) \text{tr}(\hat{f}(\gamma) \hat{f}(\gamma)^*) \right)^{1/2} \leq |E|_2^{2\beta} \left(\frac{\lambda^{1/2}}{(1 - 2\beta)^{1/2}} + 1 \right) \|w^\beta f\|_2,$$

as required.

3. Compact manifolds. In this section X will denote a compact oriented riemannian manifold. A suitable reference is Warner [10]. Let d denote the metric on X induced by the given riemannian structure on X . Fix $x_0 \in X$ and write $\phi(x) = d(x, x_0)$ for $x \in X$.

Denote the Laplace-Beltrami operator (with respect to the given riemannian structure) on $C^\infty(X)$, the space of infinitely differentiable functions on X , by Δ . The spectrum Λ of Δ is of the form $\Lambda = \{\lambda_1, \lambda_2, \dots\}$, where $0 \leq \lambda_1 < \lambda_2 < \dots$. Let \mathcal{H}_λ be the eigenspace corresponding to $\lambda \in \Lambda$. Then $d(\lambda) = \dim \mathcal{H}_\lambda < \infty$ and

$$L^2(X) = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda.$$

Fix an orthonormal basis $\phi_\lambda^{(1)}, \dots, \phi_\lambda^{d(\lambda)}$ for each \mathcal{H}_λ and define $c(\lambda)$ by

$$c(\lambda) = \max\{\|\phi_\lambda^{(j)}\|_\infty : j \in \{1, \dots, d(\lambda)\}\}.$$

(For simplicity we suppress the fact that $c(\lambda)$ depends upon the chosen basis.)

For each subset $E \subseteq \Lambda$ denote the orthogonal projection of $L^2(X)$ onto $\bigoplus_{\lambda \in E} \mathcal{H}_\lambda$ by P_E . (When E is a singleton $\{\lambda\}$, denote P_E by P_λ .) Suppose $0 \leq \theta < 1/2$ and define K_θ by

$$K_\theta = \|(A \circ \phi)^{-\theta}\|_2,$$

where $A(r)$ is the volume (in the canonical riemannian measure induced by the riemannian structure) of the set $\{x \in X : d(x, x_0) \leq r\}$. As in Lemma 2.2, $K_\theta < \infty$ since $0 \leq \theta < 1/2$.

Let $f \in L^2(X)$. Then

$$\begin{aligned} \|P_\lambda f\|_2^2 &= \sum_{j=1}^{d(\lambda)} \left(\int_X f \overline{\phi_\lambda^{(j)}} \right)^2 \\ &\leq c(\lambda)^2 d(\lambda) \left(\int_X |f| \right)^2 \leq K_\theta^2 c(\lambda)^2 d(\lambda) \|(A \circ \phi)^\theta f\|_2^2. \end{aligned}$$

Hence, whenever $E \subseteq \Lambda$,

$$\begin{aligned} \|P_E f\|_2^2 &\leq K_\theta^2 \sum_{\lambda \in E} c(\lambda)^2 d(\lambda) \|(A \circ \phi)^\theta f\|_2^2 \\ &= K_\theta^2 \mu(E)^2 \|(A \circ \phi)^\theta f\|_2^2, \end{aligned}$$

where $\mu(E) = (\sum_{\lambda \in E} d(\lambda) c(\lambda)^2)^{1/2}$.

In summary, with notation as above,

$$(3.1) \quad \|P_E f\|_2 \leq K_\theta \mu(E) \|(A \circ \phi)^\theta f\|_2$$

for $f \in L^2(X)$, $E \subseteq \Lambda$ and $0 \leq \theta < 1/2$ where $K_\theta < \infty$, a local uncertainty inequality directly analogous to (1.1) and Theorem 2.4.

3.1 THE TWO-DIMENSIONAL SPHERE. Suppose $X = S^2$, the two-dimensional sphere, with the usual riemannian structure. In spherical coordinates

$$S^2 = \{(\alpha, \beta): 0 \leq \alpha < 2\pi, 0 \leq \beta \leq \pi\}$$

with the usual identifications. The eigenvalues of the Laplace-Beltrami operator are $n(n+1)$ with $n \in \mathbf{N} = \{0\} \cup \mathbf{Z}^+$ and the corresponding eigenspaces $\mathcal{H}_{n(n+1)}$ have dimension $2n+1$ [8]. Let $\{Y_n^m: -n \leq m \leq n, m \in \mathbf{Z}\}$ be the associated spherical functions: they form a column of entry functions for the usual description of the $(2n+1)$ -dimensional representation of $SU(2)$ [9] and thus satisfy $\|Y_n^m\|_\infty \leq 1$.

The functions $\{(2n+1)^{1/2} Y_n^m: -n \leq m \leq n, m \in \mathbf{Z}\}$ make up an orthonormal basis for $\mathcal{H}_{n(n+1)}$ and so, with respect to this basis, $c_n = c(n(n+1)) \leq (2n+1)^{1/2}$. The metric $d(\cdot, \cdot)$ with respect to the usual riemannian structure on S^2 is just the euclidean distance along great circles. Define ϕ on S^2 by $\phi(\alpha, \beta) = \beta$, that is, the geodesic distance between the pole $(0, 0)$ and (α, β) . Then

$$(A \circ \phi)(\alpha, \beta) = 2\pi(1 - \cos \beta),$$

the surface area of the cap $\{(\alpha, \beta'): 0 \leq \alpha < 2\pi, 0 \leq \beta' \leq \beta\}$. Suppose $E \subseteq N$; with the above notation, (3.1) becomes

$$\|P_E f\|_2 \leq K_\theta \left(\sum_{n \in E} (2n+1)^2 \right)^{1/2} \int_{S^2} (2\pi(1 - \cos \beta))^{2\theta} |f(\alpha, \beta)|^2 d\mu$$

for $f \in L^2(S^2)$ and $0 \leq \theta < 1/2$ where

$$\begin{aligned} K &= \left(\int_{S^2} |A \circ \phi|^{-2\theta} \right)^{1/2} \\ &= \int_0^{2\pi} \int_0^\pi (2\pi(1 - \cos \beta))^{-2\theta} \sin \beta d\beta d\alpha \\ &= 2(2\pi)^{-2\theta} (1 - 2\theta)^{-1} \end{aligned}$$

and $d\mu$ is the (riemannian) measure given by $d\mu = \sin \beta d\beta d\alpha$.

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