

THE DENSITY OF PEAK POINTS IN THE SHILOV BOUNDARY OF A BANACH FUNCTION ALGEBRA

TAHER G. HONARY

(Communicated by John B. Conway)

ABSTRACT. H. G. Dales has proved in [1] that if A is a Banach function algebra on a compact metrizable space X , then $\overline{S_0}(A, X) = \Gamma(A, X)$, where $S_0(A, X)$ is the set of peak points of A (w.r.t. X) and $\Gamma(A, X)$ is the Shilov boundary of A (w.r.t. X). Here, by considering the relation between peak sets and peak points of a Banach function algebra A and its uniform closure \overline{A} , we present an elementary and constructive proof of the density of peak points in the Shilov boundary.

Introduction. For the notations, definitions, elementary and known results, one can refer to [2].

Let X be a compact Hausdorff space and let $C(X)$ denote the space of all continuous complex valued functions on X . A function algebra on X is a subalgebra of $C(X)$ which contains the constants and separates the points of X . Now consider a norm for the function algebra A on X such that $\|f \cdot g\| \leq \|f\| \cdot \|g\|$ for all $f, g \in A$. A complete normed function algebra on X is called a Banach function algebra on X . If the norm of a Banach function algebra is the uniform norm on X , i.e., if $\|f\|_X = \sup_{x \in X} |f(x)|$, it is called a uniform algebra on X . If A is a function algebra on X , then \overline{A} , the uniform closure of A , is a uniform algebra on X .

If $(A, \|\cdot\|)$ is a Banach function algebra on X , for every $x \in X$ the map $\phi_x: A \rightarrow \mathbb{C}$, defined by $\phi_x(f) = f(x)$, is a nonzero continuous complex homomorphism on A and so $\phi_x \in M_A$, where M_A is the maximal ideal space of A . Clearly for every $f \in A$,

$$\|f\|_X = \sup_{x \in X} |f(x)| = \sup_{x \in X} |\phi_x(f)| \leq \sup_{\phi \in M_A} |\phi(f)| \leq \|f\|.$$

Let A be a function algebra on X . A closed subset P of X is called a peak set of A (w.r.t. X) if there exists a function $f \in A$ such that $f = 1$ on P and $|f| < 1$ on $X \setminus P$. If $P = \{p\}$, then p is called a peak point of A (w.r.t. X) and the set of all peak points of A (w.r.t. X) is denoted by $S_0(A, X)$. A subset E of X is called a boundary for A (w.r.t. X) if every function $f \in A$ attains its maximum modulus on E . Evidently every boundary must contain $S_0(A, X)$. The Shilov boundary of A (w.r.t. X) is the smallest closed boundary of A (w.r.t. X) and it is denoted by $\Gamma(A, X)$. It is well known that the Shilov boundary for function algebras exists and it is in fact the intersection of all closed boundaries.

A local characterization of the points of $\Gamma(A, X)$ is clear: A point $x \in X$ is in $\Gamma(A, X)$ if and only if for any neighbourhood N_x of x there exists $f \in A$ such that

Received by the editors February 6, 1987 and, in revised form, May 6, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 46J10, 46J20.

©1988 American Mathematical Society
0002-9939/88 \$1.00 + \$.25 per page

the set on which f attains its maximum modulus is contained in N_x , or, in other words, there exists a peak set P of A such that $P \subseteq N_x$.

It is a well-known theorem of Bishop that for a uniform algebra A on a compact metrizable space X every peak set of A contains a peak point of A and so the set of peak points of A is a boundary and therefore is dense in the Shilov boundary of A , i.e., $\overline{S}_0(A, X) = \Gamma(A, X)$.

H. G. Dales has extended this result in [1] to any Banach function algebra A on a compact metrizable space X . Now we give another proof of the density of peak points in the Shilov boundary of a Banach function algebra which is constructive and elementary.

THEOREM. *If A is a Banach function algebra on a compact metrizable space X , then the set of peak points of A is dense in the Shilov boundary of A , i.e., $\overline{S}_0(A, X) = \Gamma(A, X)$.*

PROOF. It is sufficient to show that any neighbourhood N_0 of a peak set P_0 of A contains a peak point of A .

If P_0 is a peak set of A it is also a peak set of \overline{A} , and so it contains a peak point p_1 of \overline{A} . Thus there exists $f_1 \in \overline{A}$ such that $f_1(p_1) = 1$ and $|f_1| < 1$ on $X \setminus \{p_1\}$. Now let N_1 be a neighbourhood of p_1 such that $\overline{N}_1 \subseteq N_0$. Take $\beta_1 = \max_{x \in X \setminus N_1} |f_1(x)|$. Then $\beta_1 < 1$. Given $\varepsilon_1 > 0$ there exists $g_1 \in A$ such that $\|g_1 - f_1\|_X < \varepsilon_1$ and so $|g_1(p_1)| - |g_1(q)| > 1 - 2\varepsilon_1 - \beta_1$ for $q \in X \setminus N_1$. By taking ε_1 small enough so that $1 - 2\varepsilon_1 - \beta_1 > 0$ we have $|g_1(q)| < |g_1(p_1)|$ for $q \in X \setminus N_1$. Thus g_1 attains its maximum modulus within N_1 .

If $g_0 \in A$ peaks on P_0 , then

$$|g_0| + \eta_1 |g_1| < 1 + \eta_1(\varepsilon_1 + \beta_1) \quad \text{on } X \setminus N_1$$

and

$$|g_0(p_1)| + \eta_1 |g_1(p_1)| > 1 + \eta_1(1 - \varepsilon_1) \quad \text{for every } \eta_1 > 0.$$

Hence

$$|g_0(p_1)| + \eta_1 |g_1(p_1)| - |g_0 - \eta_1 |g_1|| > \eta_1(1 - 2\varepsilon_1 - \beta_1) > 0 \quad \text{on } X \setminus N_1.$$

Thus $|g_0| + \eta_1 |g_1|$ attains its maximum within N_1 . Clearly we can find $\phi_1 (0 \leq \phi_1 \leq 2\pi)$ such that $G_1 = g_0 + \eta_1 e^{i\phi_1} g_1$ attains its maximum modulus within N_1 . In fact

$$\|G_1\|_X - |G_1(q)| > \eta_1(1 - 2\varepsilon_1 - \beta_1) > 0 \quad \text{for } q \in X \setminus N_1.$$

Since $G_1 \in A$, the maximum set of G_1 contains a peak point $p_2 \in N_1$ of \overline{A} and so there exists $f_2 \in \overline{A}$ such that $f_2(p_2) = 1$ and $|f_2| < 1$ on $X \setminus \{p_2\}$. As before let N_2 be a neighbourhood of p_2 such that $\overline{N}_2 \subset N_1$ and take $\beta_2 = \max_{x \in X \setminus N_2} |f_2(x)|$ so that $\beta_2 < 1$. For $\varepsilon_2 > 0$ small enough, there exists $g_2 \in A$ which attains its maximum modulus within N_2 . Given $\eta_2 > 0$, as before we can construct $G_2 \in A$ such that $G_2 = g_0 + \eta_1 e^{i\phi_1} g_1 + \eta_2 e^{i\phi_2} g_2$ attains its maximum modulus within N_2 and

$$\|G_2\|_X - |G_2(q)| > \eta_2(1 - 2\varepsilon_2 - \beta_2) > 0 \quad \text{for } q \in X \setminus N_2.$$

Now we continue in this way to get a sequence of neighbourhoods $\{N_n\}_1^\infty$ such that $\overline{N}_n \subseteq N_{n-1}$ ($n = 1, 2, \dots$) and

$$G_n = g_0 + \eta_1 e^{i\phi_1} g_1 + \dots + \eta_n e^{i\phi_n} g_n \in A$$

attains its maximum modulus within N_n . Moreover for each n ,

$$\|G_n\|_X - |G_n(q)| > \eta_n(1 - 2\varepsilon_n - \beta_n) > 0 \quad \text{for } q \in X \setminus N_n$$

and so $\|G_n\|_X \geq \sup_{q \in X \setminus N_n} |G_n(q)| + \eta_n(1 - 2\varepsilon_n - \beta_n)$.

Next take $\Delta_n = \|G_n\|_X - \sup_{q \in X \setminus N_n} |G_n(q)|$ and define the sequence $\{\delta_n\}_1^\infty$ by $\delta_n = \min\{1/n, \Delta_1, \Delta_2, \dots, \Delta_n\}$ and $g = g_0 + \sum_{n=1}^\infty \eta_n e^{i\phi_n} g_n$ by choosing η_n small enough such that this series is convergent in the norm of A . To show this let $g = G_n + H_n$ and suppose $\eta_1, \eta_2, \dots, \eta_{k-1}$ has been chosen so that G_1, G_2, \dots, G_{k-1} and hence δ_{k-1} are determined and define $\eta_k = \delta_{k-1}/2^k \|g_k\|$. Since $\delta_k \leq \delta_n$ for $k \geq n$ we have

$$\|H_n\|_X \leq \|H_n\| \leq \sum_{k=n+1}^\infty \eta_k \|g_k\| = \sum_{k=n+1}^\infty \frac{\delta_{k-1}}{2^k} \leq \delta_n \sum_{k=n+1}^\infty \frac{1}{2^k} < \frac{\delta_n}{2},$$

but $\delta_n \xrightarrow{n \rightarrow \infty} 0$ and so $\|H_n\| \xrightarrow{n \rightarrow \infty} 0$, i.e., the series $\sum_{n=1}^\infty \eta_n e^{i\phi_n} g_n$ is convergent in norm of A . Therefore $g \in A$.

Let $p_n \in N_n$ ($n = 1, 2, \dots$) such that $|G_n(p_n)| = \|G_n\|_X$. Then

$$|g(p_n)| \geq |G_n(p_n)| - |H_n(p_n)| > \|G_n\|_X - \|H_n\|_X > \|G_n\|_X - \delta_n/2$$

and so

$$\begin{aligned} |g(q)| &\leq |G_n(q)| + |H_n(q)| \leq \sup_{q \in X \setminus N_n} |G_n(q)| + \|H_n\|_X \\ &< \|G_n\|_X - \Delta_n + \delta_n/2 \leq \|G_n\|_X - \delta_n/2 < |g(p_n)| \end{aligned}$$

for $q \in X \setminus N_n$. Therefore g attains its maximum modulus within N_n for each n and so $M_g = \{x \in X : \|g\|_X = |g(x)|\} \subseteq \bigcap_{n=1}^\infty N_n$.

Since X is compact it has the finite intersection property and so $\bigcap_{n=1}^\infty \overline{N_n} \neq \emptyset$. If we choose the sequence $\{N_n\}_1^\infty$ in such a way that the diameter of N_n approaches zero, then $\bigcap_{n=1}^\infty N_n$ must be a singleton and thus a peak point for A . This completes the proof of the theorem.

Special thanks are due to the University for Teacher Education in Tehran who supported me financially throughout this work.

REFERENCES

1. H. G. Dales, *Boundaries and peak points of Banach function algebras*, Proc. London Math. Soc. **21** (1971), 121-136.
2. C. E. Rickart, *The general theory of Banach algebras*, Van Nostrand, Princeton, N. J., 1960.

DEPARTMENT OF MATHEMATICS, UNIVERSITY FOR TEACHER EDUCATION, TEHRAN
15614 I. R. IRAN