PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 2, June 1988

HYPERBOLIC LENGTHS OF GEODESICS SURROUNDING TWO PUNCTURES

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(Communicated by Irwin Kra)

ABSTRACT. For the plane regions $\Omega_1 = \{|z| < R, z \neq 0, 1\}$ with $R > 1$, and $\Omega_2 = \mathbf{C} \setminus \{0, 1, p\}$ with $|p| = R > 1$, we describe, as $R \to \infty$, the hyperbolic lengths of the geodesics surrounding 0 and 1. Upper and lower bounds for the lengths are also stated, and these results are used to obtain inequalities, which are precise in a certain sense, for the length of the geodesic surrounding 0 and 1 in an arbitrary plane region Ω satisfying $\Omega_1 \subset \Omega \subset \Omega_2$.

1. Introduction. Suppose Ω is a region in the extended complex plane with at least three boundary points. By the Uniformization Theorem there exists a conformal universal covering λ of Ω by the upper half-plane $U = \{\operatorname{Im} \tau > 0\}$, and hence a hyperbolic structure on Ω with metric $\rho(z)|dz|$ defined by

$$\rho(z)|dz| = (\operatorname{Im} \tau)^{-1}|d\tau|,$$

where $z = \lambda(\tau)$.

In this paper we assume

(i) Ω contains all points in the disc $\{|z| < R\}$, with $R > 1$, except for 0 and 1,

(ii) Ω does not contain the point at ∞,

(iii) the complement of Ω contains at least one point p with $|p| = R$.

Our purpose is to describe the behaviour as $R \to \infty$ of the hyperbolic length L of the (unique) geodesic in $\mathscr{F}$, the homotopy class in Ω of a circle which separates $\{0, 1\}$ from $\{|z| = R\}$. The determination of L is of interest, not only in its own right, but also because it is related to trace T, the trace of the hyperbolic covering transformation T of λ determined by $\mathscr{F}$, according to

$$|\operatorname{trace} T| = 2\cosh(L/2).$$

For the two extreme regions which satisfy conditions (i), (ii), (iii), we obtain the following results.

THEOREM 1. *Suppose*

$$\Omega_1 = \Omega_1(R) = \{|z| < R, z \neq 0, 1\}.$$

Then as $R \to \infty$ the hyperbolic length L_1 of the geodesic in the homotopy class in Ω_1 of a circle separating $\{0, 1\}$ from $\{|z| = R\}$ satisfies

$$L_1 = \frac{2\pi^2}{\log 16R + o(1)}. \tag{1}$$

Received by the editors March 19, 1986 and, in revised form, March 31, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30C75, 51M25.

THEOREM 2. *Suppose*

$$\Omega_2 = \mathbf{C} \setminus \{0, 1, p \textit{ with } |p| = R\}.$$

Then the hyperbolic length L_2 of the similarly defined geodesic in Ω_2 satisfies

$$L_2 = \frac{2\pi^2}{\log 256R + o(1)}. \tag{2}$$

REMARK. A straightforward application of Ahlfors' Comparison Principle for hyperbolic densities (Ahlfors [**1**, p. 13]) shows that if Ω is any region satisfying conditions (i), (ii), (iii), the hyperbolic length L of the geodesic in the homotopy class in Ω of a circle separating $\{0, 1\}$ from $\{|z| = R\}$ satisfies

$$\frac{2\pi^2}{\log 256R + o(1)} \leq L \leq \frac{2\pi^2}{\log 16R + o(1)}. \tag{3}$$

Further, the coefficients 256 and 16 in (3) cannot be replaced by better ones in any inequalities of similar form.

2. Proof of the theorems.

PROOF OF THEOREM 1. We recall that the classical elliptic modular function $\lambda(\tau)$ maps U onto $\mathbf{C} \setminus \{0, 1\}$ in such a way that $\tau = 0, 1, \infty$ correspond, respectively, to $z = 1, \infty, 0$. Furthermore, λ is automorphic with respect to the group of elements T of the form

$$T(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \text{where } a, d \text{ are odd, } b, c \text{ are even, and } ad - bc = 1. \tag{4}$$

By the mapping λ a circle $\{|z| = r < 1\}$ is the image of an arc Γ_r which joins the lines $\{\operatorname{Re} \tau = 0\}$ and $\{\operatorname{Re} \tau = 2\}$, and which is contained within the doubly indented half-strip

$$\Delta = \{\operatorname{Im} \tau > 0\} \cap \{0 < \operatorname{Re} \tau < 2\} \cap \{|\tau - \tfrac{1}{2}| > \tfrac{1}{2}\} \cap \{|\tau - \tfrac{3}{2}| > \tfrac{1}{2}\}.$$

From the infinite product representation (Nehari [**2**, p. 319]),

$$z = \lambda(\tau) = 16q \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8, \qquad q = \varepsilon^{\pi i \tau},$$

it follows that for $\tau \in \Gamma_r$ we have

$$\operatorname{Im} \tau = (1/\pi) \log(16/r) + o(1) \tag{5}$$

as $r \to 0$.

Now define the associated function Λ by

$$\Lambda(\tau) = (\lambda(\tau + 1))^{-1},$$

so Λ is a conformal universal covering of $\mathbf{C} \setminus \{0, 1\}$ by U, with $\tau = 0, 1, \infty$ corresponding, respectively, to $z = 0, 1, \infty$. The circle $\{|z| = R > 1\}$ is then the image under Λ of an arc C_R which consists of an arc in Δ which joins $\{\operatorname{Re} \tau = 0\}$ and $\{\operatorname{Re} \tau = 2\}$, together with all its translates by integer multiples of 2. If $\tau \in C_R$ it follows from (5) that

$$\operatorname{Im} \tau = (1/\pi) \log 16R + o(1) \tag{6}$$

as $R \to \infty$.

Suppose that $|\Lambda(\tau)| = R$ and $\tau \notin C_R$. Then there exists a Möbius transformation T of the form (4) so that $T(\tau) \in C_R$. Further, T is not a translation and so $|c| \geq 2$. Consequently

$$\operatorname{Im} T(\tau) = \frac{\operatorname{Im}\tau}{|c\tau + d|^2} \leq \frac{1}{4\operatorname{Im}\tau},$$

and so by (6) we obtain

$$\operatorname{Im}\tau \leq \frac{1}{4\operatorname{Im} T(\tau)} = \frac{\pi}{4\log 16R + o(1)}. \tag{7}$$

Under the mapping Λ, Ω_1 is the image of a strip-like region S, bounded by C_R and a collection of curves $\{T_i(C_R)\}$ where the T_i are not translations. From (6) and (7) it follows that

$$S_1 \subset S \subset S_2, \tag{8}$$

where

$$S_1 = \left\{\frac{\pi}{4\log 16R + o(1)} < \operatorname{Im}\tau < \pi^{-1}\log 16R + o(1)\right\},$$
$$S_2 = \{0 < \operatorname{Im}\tau < \pi^{-1}\log 16R + o(1)\}.$$

Consider the geodesics, with respect to the respective hyperbolic densities, in the families of arcs which join the verticals $\{\operatorname{Re}\tau = 0\}$ and $\{\operatorname{Re}\tau = 2\}$ in S_1, S, and S_2. The hyperbolic density of the strip $\{0 < \operatorname{Im}\tau < w\}$ is

$$\rho(\tau) = \pi w^{-1}\csc[\pi w^{-1}\operatorname{Im}\tau],$$

and so the length of the geodesic joining verticals at Euclidean distance 2 is $2\pi w^{-1}$. Consequently the geodesic under consideration in S_1 and S_2 both have length $2\pi^2/(\log 16R + o(1))$. In S the geodesic has length L_1, and so by Ahlfors' Comparison Principle and (8) we obtain (1).

REMARK. By making a more careful analysis of the behaviour of the elliptic modular function on the arc Γ_r, the above method can be made to yield the inequalities

$$\frac{2\pi^2}{\log(16R + 8)} < L_1 < \frac{2\pi^2}{\log(16(R^2 - R)^{1/2}) - \pi^2(4\log(16(R^2 - R)^{1/2}))^{-1}}, \tag{9}$$

where the second inequality holds provided the denominator on the right is positive.

PROOF OF THEOREM 2. Let $\tau_0 \in C_R$ be such that $\Lambda(\tau_0) = p$. Ω_2 is the image under Λ of the complement K in U of a row of points $\tau_n = \tau_0 + 2n$, $n \in \mathbf{Z}$, and the collection $\{T_i(\tau_0)\colon T_i \text{ not a translation}\}$. As in the proof of Theorem 1 we have $K_1 \subset K \subset K_2$, where

$$K_1 = \left\{\operatorname{Im}\tau > \frac{\pi}{4\log 16R + o(1)}, \tau \neq \tau_n\ \forall n \in \mathbf{Z}\right\},$$
$$K_2 = \{\operatorname{Im}\tau > 0, \tau \neq \tau_n\ \forall n \in \mathbf{Z}\}.$$

In K we consider the class $\mathscr{C}$ of arcs joining the points $\operatorname{Re}\tau_0 + is$ and $\operatorname{Re}\tau_0 + 2 + is$, where $0 < s < \operatorname{Im}\tau_0$ and the 2-periodic continuation of each arc separates the row $\{\tau_n\}$ from the collection $\{T_i(\tau_0)\}$. The hyperbolic geodesic in $\mathscr{C}$ is mapped by Λ onto, and has the same length as, the geodesic in the statement of the theorem.

The classes $\mathscr{C}_1$ and $\mathscr{C}_2$ in K_1 and K_2 are defined in an obvious similar manner. Now, the function $\exp[\pi i(\tau - \tau_0)]$ maps K_1 onto $\Omega_1(R_1)$ with

$$R_1 = \exp\left[\pi\left(\operatorname{Im}\tau_0 - \frac{\pi}{4\log 16R + o(1)}\right)\right],$$

and K_2 onto $\Omega_1(R_2)$ with $R_2 = \exp(\pi \operatorname{Im}\tau_0)$. In each case the geodesic in $\mathscr{C}_i$ is mapped onto the geodesic in $\Omega_1(R_i)$ of Theorem 1. We deduce from Ahlfors' Comparison Principle and Theorem 1 that

$$\frac{2\pi^2}{\log 16R_2 + o(1)} \leq L_2 \leq \frac{2\pi^2}{\log 16R_1 + o(1)},$$

and then (2) follows from (6).

REMARK. Again a more careful analysis can be made to yield the inequality

$$\frac{2\pi^2}{\log(256R + 136)} < L_2. \tag{10}$$

(9) and (10) then show that (3) can be replaced by

$$\frac{2\pi^2}{\log(256R + 136)} < L < \frac{2\pi^2}{\log(16(R^2 - R)^{1/2}) - \pi^2(4\log(16(R^2 - R)^{1/2}))^{-1}}.$$

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