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WHEN DO SOBOLEV SPACES FORM A HILBERT SCALE?

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ABSTRACT. In this paper we show that the usual Sobolev spaces $(H^s(\Omega))_{s\in\mathbf{R}}$ are no Hilbert scale in the sense of Krein-Petunin, if Ω is an open bounded subset of $\mathbf{R}^n$.

1. Introduction. Let Ω be an arbitrary open subset of $\mathbf{R}^n$. Then the Sobolev spaces $H^m(\Omega)$ $(m \in N)$ are usually defined by

$$H^m(\Omega) = \{u|D^\alpha u \in L^2(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \leq m\}, \tag{1.1}$$

where $D^\alpha = \partial^{\alpha_1+\cdots+\alpha_n}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$, $\alpha = \{\alpha_1,\ldots,\alpha_n\}$, $|\alpha| = \alpha_1+\cdots+\alpha_n$. The derivatives $D^\alpha u$ are taken in the weak sense and $z \in L^2(\Omega) = H^0(\Omega)$ if and only if z is measurable and

$$\|z\|_0 = \left(\int_\Omega |z|^2\,dx\right)^{1/2} < \infty. \tag{1.2}$$

With the norm

$$\|u\|_m = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|_0^2\right)^{1/2}, \tag{1.3}$$

$H^m(\Omega)$ is a Hilbert space. The inner product of two elements $u, v \in H^m(\Omega)$ is given by

$$(u,v)_m = \sum_{|\alpha|\leq m} (D^\alpha u, D^\alpha v)_0. \tag{1.4}$$

For noninteger $s > 0$, $H^s(\Omega)$ can be defined by interpolation, and for real $s < 0$, $H^s(\Omega)$ is defined as the dual space of $H_0^{-s}(\Omega)$, the closure of $\mathscr{D}(\Omega)$ in $H^{-s}(\Omega)$, where $\mathscr{D}(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω (cf., e.g., [**5**]).

These Sobolev spaces play an important role in the solution of boundary value problems for (elliptic) partial differential equations (cf., e.g., [**3**, **7**]) and in regularization of linear integral equations of the first kind with differential operators (cf. [**6**, **1**]). In [**8**], Natterer proposes a new variant of Tikhonov regularization, namely Tikhonov regularization in Hilbert scales (cf. also [**1**, **2**, **9**]). One advantage of this

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approach is that one regularizes with a smooth norm, but gets convergence rates for the regularizers in a weaker norm. Another advantage is that, if the exact solution is smooth enough, one obtains higher convergence rates than with ordinary Tikhonov regularization. Hilbert scales are defined as follows (cf. [**4**]):

Let L be a densely defined selfadjoint strictly positive operator L in a Hilbert space X that fulfills $\|Lx\| \geq \|x\|$ on its domain. For $s \geq 0$ let X_s be the completion of $\bigcap_{k=0}^{\infty} D(L^k)$ with respect to the Hilbert space norm induced by the inner product

$$(x, y)_s := (L^s x, L^s y) \tag{1.5}$$

and for $s < 0$ let X_s be the dual space of X_{-s}. Then $(X_s)_{s \in \mathbf{R}}$ is called a Hilbert scale (induced by the operator L).

If one speaks of Hilbert scales one usually thinks of "the scale of" Sobolev spaces. It has been shown in [**4**] that the Sobolev spaces $H^s(\mathbf{R}^n)$ build a Hilbert scale. In this paper we show that this is no longer true for $H^s(\Omega)$, if Ω is an open *bounded* subset of $\mathbf{R}^n$. Moreover, we will see in §2 that Sobolev spaces with certain boundary conditions are a Hilbert scale.

2. Main result. Let X_1, X_2 be real Hilbert spaces with X_2 dense in X_1 and

$$\|x\|_1 \leq \|x\|_2 \quad \text{for all } x \in X_2. \tag{2.1}$$

It is well known (cf., e.g., [**5**]) that there exists a densely defined selfadjoint strictly positive operator L in X, with $D(L) = X_2$ and $\|Lx\|_1 = \|x\|_2$ for all $x \in X_2$. In the next proposition we show that there exists only one operator L with these properties and determine L from the embedding operator $i : X_2 \to X_1$.

PROPOSITION 2.1. *$L : D(L)$ $(\subset X_1) \to X_1$ is densely defined selfadjoint and strictly positive with*

$$\|Lx\|_1 \geq \|x\|_1 \quad \textit{for all } x \in D(L) \tag{2.2}$$

such that

$$D(L) = X_2 \quad \textit{and} \quad (Lx, Ly)_1 = (x, y)_2 \quad \textit{for all } x, y \in X_2 \tag{2.3}$$

if and only if

$$L = (i^*)^{-1/2}, \tag{2.4}$$

where i: $X_2 \to X_1$ is the embedding operator and i^* is the adjoint from $X_1 \to X_2$. Since i^* is selfadjoint from $X_1 \to X_1$, $(i^*)^{1/2}$ makes sense.

PROOF. "$\Rightarrow$". Since L is selfadjoint, L is a closed operator (cf., e.g., [**7**]). Together with (2.2) this implies that $\mathbf{R}(L)$ is closed. Now (2.2) and the Fredholm Alternative $(\overline{\mathbf{R}(L)} = N(L^*)^{\perp} = N(L)^{\perp})$ imply that $\mathbf{R}(L) = X_1$. Hence, $L^{-1} : X_1 \to X_1$ and $\mathbf{R}(L^{-1}) = X_2$. With (2.1) and (2.3) we get $\|L^{-1}x\|_1 \leq \|L^{-1}x\|_2 = \|x\|_1$. Together with $(L^{-1}x, y)_1 = (L^{-1}x, LL^{-1}y)_1 = (LL^{-1}x, L^{-1}y)_1 = (x, L^{-1}y)_1$ for all $x, y \in X_1$ we get that L^{-1} is bounded and selfadjoint. Therefore, L^{-2} is bounded and selfadjoint from $X_1 \to X_1$. But for all $x \in X_2$

$$(x, L^{-2}y)_2 = (Lx, L^{-1}y)_1 = (x, y)_1 = (ix, y)_1 = (x, i^* y)_2,$$

and hence $L^{-2} = i^*$ or $L = (i^*)^{-1/2}$.

"$\Leftarrow$". (2.1) implies that $i : X_2 \to X_1$ is bounded. Therefore, the adjoint $i^* : X_1 \to X_2$ is bounded. (2.1) and $\|i\| = \|i^*\|$ imply that

$$\|i^* x\|_1 \le \|x\|_1 \quad \text{for all } x \in X_1. \tag{2.5}$$

Together with

$$(i^*x, y)_1 = (i(i^*x), y)_1 = (i^*x, i^*y)_2 = (x, i(i^*y))_1 = (x, i^*y)_1,$$

this implies that i^* is a bounded selfadjoint operator from X_1 into X_1. Since $N(i^*) = \overline{\mathbf{R}(i)}^{\perp} = \overline{X_2}^{\perp} = X_1^{\perp} = \{0\}, i^*$ is also injective. Hence, $(i^*)^{1/2}$ is defined and also bounded selfadjoint and injective from X_1 into X_1. Now we define

$$L := (i^*)^{-1/2} \quad \text{with } D(L) = \mathbf{R}((i^*)^{1/2}). \tag{2.6}$$

Since $(i^*)^{1/2}$ is injective, $\overline{D(L)} = \overline{\mathbf{R}((i^*)^{1/2})} = N((i^*)^{1/2})^{\perp} = \{0\}^{\perp} = X_1$. This means that $D(L)$ is dense in X_1. Since $(i^*)^{1/2}$ is selfadjoint from X_1 into X_1, this implies that L is a densely defined selfadjoint closed operator in X_1 (cf., e.g., [**7**]). Now (2.5) implies that $\|(i^*)^{1/2}x\|_1^2 = (i^*x, x)_1 \le \|i^*x\|_1 \cdot \|x\|_1 \le \|x\|_1^2$ and hence $\|(i^*)^{1/2}x\|_1 \le \|x\|_1$ for all $x \in X_1$. Together with (2.6) we now obtain $\|Lx\|_1 \ge \|x\|_1$ for all $x \in D(L)$. This means that L defined by (2.6) fulfills (2.2). It remains to be shown that L also fulfills (2.3).

Let now $x, y \in \mathbf{R}(i^*)$ and note that $\mathbf{R}(i^*) \subset D(L)$ and $R(i^*) \subset X_2$. Then

$$(x,y)_2 = (i^*(i^*)^{-1}x, y)_2 = ((i^*)^{-1}x, iy)_1 = ((i^*)^{-1/2}x, (i^*)^{-1/2}y)_1 = (Lx, Ly)_1$$

and hence

$$(x,y)_2 = (Lx, Ly)_1 \quad \text{for all } x, y \in \mathbf{R}(i^*). \tag{2.7}$$

Since L is closed, (2.2) implies that $(D(L), \|L \cdot\|_1)$ is an Hilbert space. Let $\{x_n\}$ be a sequence in $\mathbf{R}(i^*)$. Due to (2.7), $\{x_n\}$ is an X_2-Cauchy sequence iff $\{x_n\}$ is a $D(L)$-Cauchy sequence. Let now $\{x_n\}$ be a Cauchy sequence in both senses. Then there are unique elements $x \in X_2$ and $y \in D(L)$ such that $x_n \overset{X_2}{\to} x$ and $x_n \overset{D(L)}{\to} y$. $x_n \overset{X_2}{\to} x$ and (2.1) imply that $x_n \overset{X_1}{\to} x$. Since L is closed and $\{Lx_n\}$ is a Cauchy sequence in X_1 this implies that $x \in D(L)$ and $Lx_n \overset{X_1}{\to} Lx$, i.e., $x_n \overset{D(L)}{\to} x$ and hence $x = y$. Therefore, we have shown

$$\text{closure of } \mathbf{R}(i^*) \text{ in } X_2 = \text{closure of } \mathbf{R}(i^*) \text{ in } D(L). \tag{2.8}$$

Since $(Li^*x, Ly)_1 = ((i^*)^{1/2}x, (i^*)^{-1/2}y)_1 = (x,y)_1 = (x, iy)_1$, for all $x \in X_1$ and $y \in D(L)$, i^* is also the adjoint operator of the embedding operator $i : D(L) \to X_1$. Now (2.8) and the Fredholm Alternative imply that

$$\begin{aligned} X_2 = N(i|_{X_2})^{\perp} &= \text{closure of } \mathbf{R}(i^*) \text{ in } X_2 \\ &= \text{closure of } \mathbf{R}(i^*) \text{ in } D(L) = N(i|_{D(L)})^{\perp} = D(L). \end{aligned}$$

Since $X_2 = D(L)$ a continuity argument implies that (2.7) holds for all $x, y \in X_2$. $\square$

As a consequence of Proposition 2.1 we now show that the Sobolev spaces $H^s(\Omega)$ are no Hilbert scale, if Ω is bounded. We first consider the case $\Omega = (0,1)$.

COROLLARY 2.2. *Let $m \in N$ be fixed. There exists no Hilbert scale $(X_s)_{s\in\mathbf{R}}$ induced by an operator L defined in $L_2[0,1]$ such that $X_m = H^m[0,1]$ with the norm of* (1.3) *and $X_s = H^s[0,1]$* (*with norms equivalent to that of* (1.3)), *if $s \neq m$, $s \geq 0$.*

PROOF. Due to Proposition 2.1 there exists only one densely defined selfadjoint and strictly positive operator L in $L_2[0,1]$ such that $D(L) = H^m[0,1]$ and $\|Lx\|_0 = \|x\|_m$ for all $x \in H^m[0,1]$. Therefore, we only have to show that there exists an $s > 0$ such that $X_s := D(L^{s/m}) \neq H^s[0,1]$. We will show that $X_{2m} = D(L^2) = \mathbf{R}(i^*) \subsetneqq H^{2m}[0,1]$, where i^* is the adjoint of the embedding operator $i : H^m[0,1] \to L^2[0,1]$.

By definition of i^*, for all $x \in H^m[0,1]$ and $y \in L^2[0,1]$ we have

$$(x,y)_0 = \sum_{k=0}^{m} (x^{(k)}, i^* y^{(k)})_0. \tag{2.9}$$

Now let $z \in H^{2m}[0,1]$. Then it follows from integration by part that

$$\begin{aligned} \sum_{k=0}^{m} (x^{(k)}, z^{(k)})_0 = {} & \left(x, \sum_{k=0}^{m} (-1)^k z^{(2k)} \right)_0 \\ & + \sum_{l=0}^{m-1} (x^{(l)}(1) \cdot B_l z(1) - x^{(l)}(0) \cdot B_l z(0)), \end{aligned} \tag{2.10}$$

where

$$B_l z := \sum_{k=0}^{m-(l+1)} (-1)^k z^{(2k+l+1)}, \qquad l = 0,1,\ldots,m-1. \tag{2.11}$$

Now we show that there is a $z \in H^{2m}[0,1]$ such that

$$(x,y)_0 = \sum_{k=0}^{m} (x^{(k)}, z^{(k)})_0. \tag{2.12}$$

Since i^*y is unique in (2.9), $i^*y = z$. Since $H_0^m[0,1] = \{u \in H^m[0,1] \,|\, u^{(k)}(0) = 0 = u^{(k)}(1),\ k = 0,1,\ldots,m-1\}$ is dense in $L^2[0,1]$ (cf., e.g., [**5**]) (2.12) is solvable if and only if (cf. (2.10), (2.11))

$$y = \sum_{k=0}^{m} (-1)^k z^{(2k)} =: Tz. \tag{2.13}$$

From elements in $H^m[0,1] \backslash H_0^m[0,1]$ we derive the boundary conditions

$$B_l z(0) = 0 = B_l z(1), \qquad l = 0,1,\ldots,m-1. \tag{2.14}$$

It follows from [**7**, Theorem 5, p. 77] that (2.13) and (2.14) define a selfadjoint differential operator T in $L_2[0,1]$. Note that for all $z \in D(T)$ $\|z\|_0^2 \leq \|z\|_m^2 = (z,Tz)_0 \leq \|z\|_0 \cdot \|Tz\|_0$ and hence $\|z\|_0 \leq \|Tz\|_0$. But this implies that $\mathbf{R}(T) = L_2[0,1]$. Therefore, we have shown that (2.13) and (2.14) are uniquely solvable for all $y \in L_2[0,1]$. Together with (2.9)–(2.12) this implies that $z = i^*y$ is the solution of (2.13) and (2.14). Hence,

$$\mathbf{R}(i^*) = \{z \in H^{2m}[0,1] \,|\, B_l z(0) = 0 = B_l z(1),\ l = 0,1,\ldots,m-1\} \subsetneqq H^{2m}[0,1]. \qquad \square$$

Proposition 2.1 shows us that it is always possible to define a Hilbert scale $(X_s)_{s\in\mathbf{R}}$ induced by an operator L in $L_2[0,1]$ such that $X_m = H^m[0,1]$ (for any fixed m) with the same norm. It then follows from an interpolation result (cf., e.g., [**5**]) that $X_s = H^s[0,1]$ with equivalent norms for all $0 \leq s \leq m$. But Corollary 2.2 shows that for $s = 2m$ this is no longer true. One can show, using interpolation theory, that $X_s \subsetneqq H^s[0,1]$ for all $s \geq m + \frac{1}{2}$. So $(H_s[0,1])_{s\geq 0}$ is no Hilbert scale, but for any fixed $m \in N$ $(H_s[0,1])_{0\leq s\leq m}$ is part of a Hilbert scale. In Corollary 2.2 X_m had to be equal to $H^m[0,1]$ with the same norm. We did not allow X_m to have an equivalent norm. This is crucial for Tikhonov regularization in Hilbert scales, since the convergence rates results for a regularized solution of an ill-posed problem depend on the norm of the space in which one regularizes (cf., e.g., [**1**]).

We now show that an analogous result to Corollary 2.2 also holds for Sobolev spaces $H^s(\Omega)$, where Ω is an open bounded subset of $\mathbf{R}^n$, $(n > 1)$, with sufficiently smooth boundary. We only consider the case $m = 1$.

COROLLARY 2.3. *Let $n > 1$ be fixed and let Ω be an open bounded subset of $\mathbf{R}^n$ with $C^{1,1}$ boundary (i.e., the boundary is continuously differentiable and the first derivative is Lipschitz). Then there exists no Hilbert scale $(X_s)_{s\in\mathbf{R}}$ induced by an operator L defined in $L_2(\Omega)$ such that $X_1 = H^1(\Omega)$ with the norm of* (1.3) *and $X_s = H^s(\Omega)$ (with norms equivalent to that of* (1.3)), *if $s \neq 1$, $s \geq 0$.*

PROOF. Analogously to the proof of Corollary 2.2 we only have to show that $\mathbf{R}(i^*) \subsetneqq H^2(\Omega)$, where i^* is the adjoint of the embedding operator $i\colon H^1(\Omega) \to L^2(\Omega)$. By definition of i^*, for all $x \in H^1(\Omega)$ and $y \in L^2(\Omega)$ we have

$$(x,y)_0 = (x, i^*y)_0 + (\nabla x, \nabla i^* y)_0. \tag{2.15}$$

Now let $z \in H^2(\Omega)$. Since $H_0^1(\Omega)$ is dense in $L_2(\Omega)$ (cf. [**5**]), Greens identity,

$$(\nabla x, \nabla z)_0 = -(x, \Delta z)_0 + \int_\Gamma x \frac{\partial z}{\partial v}\, d\sigma, \tag{2.16}$$

where Γ is the boundary of Ω and $\partial/\partial v$ is the normal derivative, implies that $z \in H^2(\Omega)$ solves

$$(x,y) = (x,z)_0 + (\nabla x, \nabla z)_0, \tag{2.17}$$

if and only if

$$y = z - \Delta z \tag{2.18}$$

and

$$\int_\Gamma x \frac{\partial z}{\partial v}\, d\sigma = 0 \quad \text{for all } x \in H^1(\Omega). \tag{2.19}$$

The trace theorem (cf. [**3**, Theorem 1.5.1.2]) implies that $x \to x|_\Gamma$ is surjective from $H^1(\Omega) \to H^{1/2}(\Gamma)$. Since $H^{1/2}(\Gamma)$ is dense in $L_2(\Gamma)$, this implies that (2.19) is equivalent to

$$\frac{\partial z}{\partial v} = 0 \quad \text{on } \Gamma. \tag{2.20}$$

Now [**3**, Theorem 2.4.2.7] implies that (2.18) together with (2.20) has a unique solution for all $y \in L_2(\Omega)$. Therefore, (2.15) and (2.17) imply that $\mathbf{R}(i^*) = \{z \in H^2(\Omega) | \partial z/\partial v = 0\} \subsetneqq H^2(\Omega)$. □

Corollary 2.3 not only tells us that $(H^s(\Omega))_{s\geq 0}$ is no Hilbert scale. By interpolation theory, it follows from the proof of Corollary 2.3 that $(H^s(\Omega))_{0\leq s\leq 1}$ is part of a Hilbert scale $(X_s)_{s\geq 0}$, where for $s > 1$, X_s is the Sobolev space $H^s(\Omega)$ with certain boundary conditions.

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