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PRODUCTS OF COMPLETION REGULAR MEASURES

CONSTANTINOS GRYLLAKIS

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ABSTRACT. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{j\in J} Y_j$, where all X_i, Y_j are separable metric spaces. Let μ and ν be completion regular Radon probability measures on X and Y respectively. Then $\mu\times\nu$ on $X\times Y$ is completion regular.

This solves a problem of J. R. Choksi and D. H. Fremlin.

1. Introduction. A Radon measure μ on a completely regular space is completion regular iff every Borel set is measurable with respect to the completion of the Baire restriction of μ. That is, to every Borel set E there correspond two Baire sets A and B such that $A \subset E \subset B$ and $\mu(B - A) = 0$.

In [**1**] J. R. Choksi posed the problem. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{j\in J} Y_j$ where all X_i, Y_j are compact metric spaces. Let μ and ν be completion regular Radon measures on X, Y with full support. When is $\mu \times \nu$ on $X \times Y$ completion regular? See also [**3**, p. 121, note after Theorem 4]. The following partial results are known.

(a) If both μ, ν are product measures of probability measures with full support on X, Y then the answer is yes: see Kakutani [**7**].

(b) If just one of μ and ν is a product measure the answer is yes: see Choksi and Fremlin [**3**].

Also Fremlin [**6**] has shown that for arbitrary compact spaces X and Y the answer is no.

In this paper it is proved that the answer to the above problem is always yes.

Moreover, the assumption that μ and ν have full support is unnecessary and the assumption of compactness of X_i, Y_j is replaced by the more general assumption of separability.

For more information concerning the above problem as well as completion regular measures on product spaces the reader can consult [**2**].

2. In this section we will give a characterization of completion regularity of measures on uncountable products of separable metric spaces.

Let $\langle X_i\rangle_{i\in I}$ be a family of separable metric spaces and $X = \prod_{i\in I} X_i$. For every $i \in I$, let T_i be a countable base for the topology of X_i.

For $J \subset I$, $\mathrm{pr}_J\colon X \to \prod_{i\in J} X_i$ denotes the canonical projection. If $A \subset X$ has the form $A = \mathrm{pr}_J^{-1}(B)$ where $B \subset \prod_{i\in J} X_i$ and $J \subset I$, we say that A depends on (the set of coordinates) J.

A subset U of X is called elementary open (resp. basic elementary open) if it has the form $U = \mathrm{pr}_k^{-1}(\prod_{i\in k} V_i)$, where $K \subset I$ is finite and $V_i \subset X_i$ is open (resp. $V_i \in T_i$).

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The set of basic elementary open sets constitute a base for the topology of X.

PROPOSITION. *Let $\langle X_i \rangle_{i \in I}$ be an uncountable family of separable metric spaces and μ Radon probability measure on $X = \prod_{i \in I} X_i$. Then the following are equivalent*

(a) *The measure μ is completion regular,*

(b) *for every uncountable family $\langle U_\alpha \rangle_{\alpha \in \Lambda}$ of elementary open sets, if there is a pairwise disjoint family $\langle I_\alpha \rangle_{\alpha \in \Lambda}$ of finite sets of coordinates such that U_α depends on I_α then $\mu(\bigcup_{\alpha \in \Lambda} U_\alpha) = 1$, and*

(c) *for every open set $U \subset X$, there exist an $U' \supset U$ with $\mu(U') = \mu(U)$ and U' is a countable union of elementary open sets.*

PROOF. (a) $\Rightarrow$ (b) Let $\langle U_\alpha \rangle_{\alpha \in \Lambda}$ be as in (b). Since μ is completion regular, it suffices to show that if B is a Baire set and $B \supset (\bigcup_{\alpha \in \Lambda} U_\alpha)$ then B must be the whole space X.

Indeed B, as a Baire set, depends on a countable set $I^* \subset I$ (see [**8**]). Thus we can find $\gamma \in \Lambda$ with $I^* \cap I_\gamma = \varnothing$ and since $B \supset U_\gamma$, it follows that $B \supset U_\gamma^c$ so $B = X$.

(b) $\Rightarrow$ (c) Let U be an open subset of X. Then $U = \bigcup_{\alpha \in A} U_\alpha$ where $U_\alpha \neq U_\beta$ for $\alpha \neq \beta$ and each U_α is a basic elementary open set depending on a finite set $I_\alpha \subset I$. Without loss of generality, we assume that for some $k \in N$, $|I_\alpha| = k$ for every $\alpha \in A$. (If we set $V_k = \{\alpha \in A\colon |I_\alpha| = k\}$, then $U = \bigcup_{k=1}^{\infty} V_k$.)

CLAIM. For every uncountable $B \subset A$ there is an uncountable $B' \subseteq B$ and a basic elementary open set V such that V depends on less than k coordinates, $V \supset \bigcup_{\alpha \in B'} U_\alpha$ and $\mu(V) = \mu(\bigcup_{\alpha \in B'} U_\alpha)$.

First we observe that if J is a finite subset of I, then the family of all basic elementary open sets depending on J is countable. In particular, for every $\alpha \in A$, $\{\beta \in A\colon I_\beta = I_\alpha\}$ is countable, so there exists an uncountable $B_1 \subseteq B$ such that $I_\alpha \neq I_\beta$ for every $\alpha, \beta \in B_1$, $\alpha \neq \beta$.

Thus by the Erdös-Rado theorem [**4**, p. 62] there is an uncountable $B_0 \subseteq B_1$ and $J \subset I$ such that

$$I_\alpha \cap I_\beta = J \quad \text{for every } \alpha, \beta \in B_0.$$

Clearly $|J| < k$.

If $J \neq \varnothing$ then by (b) $\mu(\bigcup_{\alpha \in A} U_\alpha) = 1$ and the claim is true for $V = X$ and $B' = B$. (This is the case at least when $k = 1$.) Now assume that $J \neq \varnothing$ and choose an uncountable $B' \subset B_0$ such that

$$\mathrm{pr}_J(U_\alpha) = \mathrm{pr}_J(U_\beta) = W \quad \text{for every } \alpha, \beta \in B'.$$

Set $V = \mathrm{pr}_J^{-1}(W)$. Clearly $V \supset \bigcup_{\alpha \in B'} U_\alpha$.

For every $\alpha \in B'$ there is an elementary open set W_α such that W_α depends on $I_\alpha \backslash J$ and $U_\alpha = V \cap W_\alpha$. Since $I_\alpha \backslash J$, $\alpha \in B'$ are pairwise disjoint, (b) implies that $\mu(\bigcup_{\alpha \in B'} W_\alpha) = 1$. Thus

$$\mu\left(\bigcup_{\alpha \in B'} U_\alpha\right) = \mu\left(V \cap \bigcup_{\alpha \in B'} W_\alpha\right) = \mu(V)$$

and the proof of the claim is complete.

Using the above claim, it is easily seen (using an exhaustion argument on A) that a decreasing family $\langle B_\gamma\rangle_{\gamma<\tau_1}$ of subsets of A can be constructed by induction on the ordinal γ with the following properties.

(i) $B_0 = A$.

(ii) For every $\gamma < \tau_1$ there is a basic elementary open set V_γ^1 depending on less than k coordinates such that

$$V_\gamma^1 \supset \bigcup_{\alpha\in B_\gamma\setminus B_{\gamma+1}} U_\alpha \quad \text{and} \quad \mu(V_\gamma^1) = \mu\left(\bigcup_{\alpha\in B_\gamma\setminus B_{\gamma+1}} U_\alpha\right).$$

(iii) $B_\gamma = \bigcap_{\beta<\gamma} B_\beta$ if γ is limit.

(iv) $C_1 \equiv \bigcap_{\gamma<\tau_1} B_\gamma$ is countable.

Thus $\bigcup_{\gamma<\tau_1} V_\gamma^1 \cup (\bigcup_{\alpha\in C_1} U_\alpha) \supset \bigcup_{\alpha\in A} U_\alpha = U$ and by the regularity of μ,

$$\mu\left(\bigcup_{\gamma<\tau_1} V_\gamma^1 \cup \left(\bigcup_{\alpha\in C_1} U_\alpha\right)\right) = \mu(U).$$

If τ_1 is a countable ordinal then (c) follows. Otherwise we repeat the same argument for the family $\langle V_\gamma^1\rangle_{\gamma<\tau_1}$ in the place of $\langle U_\alpha\rangle_{\alpha\in A}$.

Thus we find a family $\langle V_\gamma^2\rangle_{\gamma<\tau_2}$ of basic elementary open sets depending on less than $k-1$ coordinates and a countable $C_2 \subset \tau_1$ such that

$$\bigcup_{\gamma<\tau_2} V_\gamma^2 \cup \left(\bigcup_{\gamma\in C_2} V_\gamma^1\right) \supset \bigcup_{\gamma<\tau_1} V_\gamma^1 \text{ and } \mu\left(\bigcup_{\gamma<\tau_2} V_\gamma^2 \cup \left(\bigcup_{\gamma\in C_2} V_\gamma^1\right)\right) = \mu\left(\bigcup_{\gamma<\tau_1} V_\gamma^1\right).$$

Again if τ_2 is countable (c) follows.

In this way, we may (if necessary) proceed at the kth step. But then $V_\gamma^k = X$ for all $\gamma < \tau_k$ and (c) follows.

(c) $\Rightarrow$ (a) obvious from the regularity of μ.

3. In this section we will prove the main result of this paper which is the following.

THEOREM. *Let $\langle X_i\rangle_{i\in I}$ and $\langle Y_j\rangle_{j\in J}$ be families of separable metric spaces and μ,ν completion regular Radon probabilities measures on $X = \prod_{i\in I} X_i$ and $Y = \prod_{j\in J} Y_j$, respectively.*

Then the product Radon measure $\mu\times\nu$ is completion regular.

For the proof of the above theorem we need a lemma.

LEMMA. *Let (X,μ) and (Y,ν) be probability measure spaces and $Q \subset X\times Y$ a countable union of measurable rectangles, i.e. $Q \subset \bigcup_{m=1}^\infty U_m \times V_m$ where each $U_m \subset X$ and $V_m \subset Y$ is measurable, with $\mu\times\nu Q < 1$.*

Then there exist a sequence $\langle B_n \times A\rangle_{n\in N}$ of measurable rectangles in $X\times Y$ with $\mu(B_n)\cdot\nu(A) > 0$, such that for every sequence of measurable rectangles $\langle B_n' \times A'\rangle_{n\in N}$ with $\mu(B_n')\nu(A') > 0$, $B_n' \subset B_n$ and $A' \subset A$, $\mu\times\nu((B_n'\times A')\setminus Q) > 0$ except for finitely many n.

PROOF. For every $C \subset X\times Y$, let C_x denote the section of C at x, i.e. $C_x = \{y \in Y\colon (x,y) \in C\}$. We set $\varepsilon = \mu\times\nu Q$ and consider a sequence $\langle\theta_n\rangle_{n\in N}$ of

positive numbers such that $\theta_1 > \varepsilon$ and $\sum_{n=1}^{\infty} \theta_n < 1$. By induction on n, we will construct two decreasing sequences $\langle B_n \rangle_{n \in N}$ and $\langle S_n \rangle_{n \in N}$ of measurable sets in X and Y, respectively, such that for every n

(i) $\mu B_n > 0$,

(ii) $\nu S_n > 1 - \sum_{m=1}^{n-1} \theta_m$ if $n \geq 2$ and $\nu S_1 = 1$,

(iii) $\nu((B_n \times S_n) \cap Q)_x \; [= \nu(S_n \cap Q_x)] < \theta_n$

for every $x \in B_n$.

For $n = 1$, take $S_1 = Y$ and $B_1 = \{x \in X\colon \nu(Q_x) < \theta_1\}$. Since $\theta_1 > \varepsilon$, applying Fubini's Theorem for Q we have $\mu B_1 > 0$.

We now suppose that $\langle B_m \rangle_{m<n}$ and $\langle S_m \rangle_{m<n}$ where $n \geq 2$ been constructed. We set

$$P_k = (B_{n-1} \times S_{n-1}) \cap \left(Q \setminus \bigcup_{m=1}^{k} (U_m \times V_m) \right)$$

and observe that $\lim_{k \to \infty} \mu \times \nu P_k = 0$.

Fix a k such that

$$(*) \qquad \mu \times \nu P_k < \theta_n \cdot \mu B_{n-1}.$$

For every $F \subset \{1, \ldots, k\}$ set

$$R_F = \{x \in B_{n-1}\colon x \in U_m \Leftrightarrow m \in F \; \forall m \in \{1, \ldots, k\}\}$$

$$\left[R_F = \left(\bigcap_{m \in F} (U_m \cap B_{n-1}) \right) \setminus \left(\bigcup_{\substack{1 \leq m \leq k_n \\ m \notin F}} (U_m \cap B_{n-1}) \right) \right]$$

$$T_F = S_{n-1} \cap \left(\bigcup_{m \in F} V_m \right) \quad \text{and}$$

$$J_F = (R_F \times (S_{n-1} \setminus T_F)) \cap Q = (R_F \times (S_{n-1} \setminus T_F)) \cap \left(\bigcup_{m > k} (U_m \times V_m) \right).$$

We observe that the set of all R_F is a finite measurable partition of B_{n-1} and the set of all J_F a finite measurable partition of P_k. Thus, if we assume that for every $F \mu \times \nu J_F \geq \theta_n \cdot \mu R_F$, then summing over F we conclude that $\mu \times \nu P_k \geq \theta_n \cdot \mu B_{n-1}$, contradicting $(*)$. Therefore there exist some F such that $\mu \times \nu J_F < \theta_n \cdot \mu R_F$. In particular, $\mu R_F > 0$.

We set $S_n = S_{n-1} \setminus T_F$.

Since $\mu \times \nu((R_F \times S_n) \cap Q) < \theta_n \cdot \mu R_F$, using Fubini's Theorem we find a measurable set $B_n \subset R_F$ such that $\mu(B_n) > 0$ and $\mu((B_n \times S_n) \cap Q)_x < \theta_n$ for every $x \in B_n$.

Let $x \in B_n$. Since $B_n \subset R_F \subset B_{n-1}$ we have $T_F \subset S_{n-1} \cap Q_x$ and, by the induction hypothesis

$$\nu(T_F) \leq \nu(S_{n-1} \cap Q_x) < \theta_{n-1} \quad \text{and} \quad \nu(S_n) = \nu(S_{n-1}) - \nu(T_F) \geq 1 - \sum_{m=1}^{n-1} \theta_m.$$

The construction of $\langle B_n \rangle_{n \in N}$ and $\langle S_n \rangle_{n \in N}$ is now complete.

We set $A = \bigcap_{n=1}^{\infty} S_n$ and prove that $\langle B_n \times A\rangle_{n\in N}$ satisfies the conclusion of the lemma. By (ii) $\nu(A) = 1 - \sum_{m=1}^{\infty} \theta_m > 0$.

Let $\langle B'_n \times A'\rangle_{n\in N}$ be a sequence of measurable rectangles such that $\mu(B'_n)\nu(A') > 0$, $B'_n \subset B_n$ and $A' \subset A$. By (iii) for every n we have

$$\nu((B'_n \times A') \cap Q)_x < \theta_n \quad \text{for every } x \in B'_n,$$

so by Fubini's Theorem,

$$\mu \times \nu((B'_n \times A') \cap Q) < \theta_n \cdot \mu(B'_n).$$

If n is such that $\mu(B'_n \times A' \backslash Q) = 0$, then

$$\mu(B'_n)\nu(A') = \mu \times \nu(B'_n \times A') = \mu \times \nu((B'_n \times A') \cap Q) < \theta_n \cdot \mu(B'_n)$$

so $\nu(A') < \theta_n$. But this can happen for only a finite number of n's.

Note. It is interesting to compare the above lemma with the following fact proved in [**5**].

THEOREM (ERDÖS-OXTOBY). *Let (X, μ) and (Y, ν) be finite diffuse measure spaces and $\varepsilon > 0$. Then there exist a sequence $\langle I_n \times J_n\rangle_{n\in N}$ of measurable rectangles such that*

(i) $\sum_{n\in N} \mu I_n \nu J_n \leq \varepsilon$.

(ii) *If $E \times F$ is a measurable rectangle with $\mu E \cdot \nu F > 0$, then there is an $n \in N$ such that $\mu(E \cap I_n)\nu(F \cap J_n) > 0$, i.e. $\mu \times \nu(Q \cap (E \times F)) > 0$ where $Q = \bigcup_{n\in N} I_n \times J_n$. (For a simple proof due to R. O. Davis see* [**6**].)

The preceding lemma says that we cannot have an extension of Erdös-Oxtoby theorem in a certain sense.

We now come to the proof of the main theorem.

PROOF. We suppose that both I and J are uncountable since if one of I, J is countable it is quite easy to prove our theorem. It will be enough to prove that $(X \times Y, \mu \times \nu)$ satisfies (b) of proposition of §2.

We suppose if possible otherwise. Then there exists a family $\langle W_\alpha\rangle_{\alpha<\omega_1}$ of elementary open sets in $X \times Y$ such that

(i) $W_\alpha = U_\alpha \times V_\alpha$, where U_α, V_α are elementary open sets in X and Y, respectively, U_α depends on a finite set $I_\alpha \subset I$, V_α on a finite set $J_\alpha \subset J$ and for every $\alpha \neq \beta$ $(I_\alpha \cup J_\alpha) \cap (I_\beta \cup J_\beta) = \varnothing$ and

(ii) $\mu \times \nu(\bigcup_{\alpha<\omega_1} W_\alpha) < 1$.

Because $\mu\times\nu$ is Radon there exists a countable $M \subset \omega_1$ such that $\mu\times\nu(\bigcup_{\alpha<\omega_1} W_\alpha) = \mu \times \nu(\bigcup_{\alpha\in M} W_\alpha)$.

Let $\langle B_n \times A\rangle_{n\in N}$ be a sequence of measurable rectangles in $X \times Y$ as in the statement of the lemma for $Q = \bigcup_{\alpha\in M} W_\alpha$.

CLAIM 1. There exists an $\alpha < \omega_1$ such that

$$\nu(V_\gamma \cap A) > 0 \quad \text{for every } \gamma > \alpha.$$

Indeed, supposing otherwise, there exists an uncountable $\Delta \subset \omega_1$ such that $\nu(V_\gamma \cap A) = 0$ for every $\gamma \in \Delta$. Because ν is Radon there exist a countable $\Delta' \subset \Delta$ such that $\nu(\bigcup_{\gamma\in\Delta} V_\gamma) = \nu(\bigcup_{\gamma\in\Delta'} V_\gamma)$ but $\nu(V_\gamma \cap A) = 0$ $\forall\gamma \in \Delta'$ so $\nu(\bigcup_{\gamma\in\Delta} V_\gamma) = \nu(\bigcup_{\gamma\in\Delta'} V_\gamma) \leq 1 - \nu A < 1$ which contradicts proposition of §2 because ν is completion regular.

For similar reasons the following claim is valid.

CLAIM 2. For every $n \in N$ there exists a $\beta_n < \omega_1$ such that $\mu(U_\gamma \cap B_n) > 0$ for every $\gamma > \beta_n$.

We choose a $\delta > \sup\{\alpha, \beta_n, n \in N\}$, $\delta < \omega$, and set $A' =: A \cap U_\delta$ and $B'_n = B_n \cap V_\delta$ for every $n \in N$. Then $\mu(B'_n) \cdot \nu(A') > 0$, $B'_n \times A' \subset B_n \times A$ and $B'_n \times A' \subset U_\delta \times V_\delta \subset \bigcup_{\alpha<\omega_1} W_\alpha$ for every n so $\mu \times \nu(B'_n \times A' \backslash Q) = 0$ for every n. But this is impossible by the choice of $\langle B_n \times A \rangle_{n \in N}$. This ends the proof of our theorem.

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