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A NOTE ON PALEY-WIENER-ZYGMUND STOCHASTIC INTEGRALS

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ABSTRACT. A Fubini theorem for multiparameter Paley-Wiener-Zygmund stochastic integrals is established which unifies various stochastic integration formulas. We also obtain similar results for double stochastic integrals of Paley-Wiener-Zygmund type.

1. Introduction. Paley-Wiener-Zygmund (P.W.Z.) stochastic integrals have been used in several recent papers, including [**4–12**, **17–20**, **28**], concerning Feynman integration theory. In particular, P.W.Z. integrals are used in the definition of a Banach algebra S of functions on Wiener space that was introduced by Cameron and Storvick [**4**]. In [**17**] Johnson showed that S was isometrically isomorphic to a Banach algebra of Fresnel integrable functions as given by Albeverio and Høegh-Krohn [**1**]. The initial motivation for this paper was a need in [**10 and 28**] for a Fubini-type stochastic integration formula involving a mix of a one-parameter Wiener process and a multiparameter Wiener process.

For $Q = [O,T]^N$, $N = 1,2,\ldots$, let $W = \{W(t): t \in Q\}$ be the standard N-parameter Wiener process. Let $\{\alpha_k\}$ be a complete orthonormal (CON) set of functions of bounded variation in the sense of Hardy-Krause [**3**] on Q. For h in $L_2(Q)$ let

$$h_n(t) = \sum_{k=1}^{n} (h, \alpha_k)\alpha_k(t).$$

Then the P.W.Z. integral is defined by the formula

$$\int_Q h(t)d^*W(t) = \underset{n\to\infty}{\text{l.i.m.}} \int_Q h_n(t)dW(t)$$

where "l. i. m." means "limit in the mean" (or limit in L_2-sense) and $\int_Q h_n(t)dW(t)$ is the ordinary Riemann-Stieltjes integral.

It is well known that for each h in $L_2(Q)$, the P.W.Z. $\int_Q h(t)d^*W(t)$ exists almost surely and is essentially independent of the CON set $\{\alpha_K\}$. Also if h is of bounded variation on Q then $\int_Q h(t)d^*W(t)$ equals the R-S integral $\int_Q h(t)dW(t)$ almost surely. Some more general P.W.Z. stochastic integrals are considered in Park [**25**] and Kuo and Russek [**22**].

For a nonanticipating function $h(t,\omega)$ in $L_2(Q \times \Omega)$, where $(\Omega, \mathscr{F}, P)$ is a probability space, Itô's stochastic integral $\int_Q h(t,\omega)dW(t)$ is extremely well known and

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widely used. In §2 of this paper we establish some conditions under which the Itô stochastic integral and the P.W.Z. stochastic integral agree almost surely. We then proceed to develop a Fubini theorem for stochastic integrals which unifies several integration formulas.

In §3 we define a P.W.Z. type double stochastic integral and show that it is equal almost surely to certain iterated stochastic integrals. We then establish a Fubini-type theorem which allows the interchange of the order of the stochastic processes and point out that a corresponding result is not possible for Itô integrals.

2. Stochastic integrals with respect to two independent processes. In our first theorem we show that under certain conditions the Itô and P.W.Z. stochastic integrals agree almost surely.

THEOREM 1. *Let W and X be two independent N-parameter Wiener processes defined on $Q = [O,T]^N$. Let $h(t,u)$ be a function on $Q \times \mathbf{R}$ such that $\int_Q E|h(t,X(t))|^2 dt < \infty$. Then*

$$\int_Q h(t,X(t))d^*W(t) = \int_Q h(t,X(t))dW(t)$$

almost surely, where the right-hand side is the Itô stochastic integral.

PROOF. Let$\{\alpha_k(t)\}_{k=1}^{\infty}$ be a CON set on Q with each α_k of bounded variation. Then

$$\begin{aligned}
&E\left[\sum_{k=1}^{n}\int_Q h(t,X(t))\alpha_k(t)dt\int_Q \alpha_k(s)dW(s) - \int_Q h(t,X(t))dW(t)\right]^2 \\
&\quad = E\left[\sum_{k=1}^{n}\left\{\int_Q h(t,X(t))\alpha_k(t)dt\right\}^2\right] + \int_Q E|h(t,X(t))|^2 dt \\
&\qquad - 2E\left[\sum_{k=1}^{n}\int_Q h(t,X(t))\alpha_k(t)dt\int_Q \alpha_k(s)dW(s)\int_Q h(t,X(t))dW(t)\right] \\
&\quad = E\left[\sum_{k=1}^{n}\left\{\int_Q h(t,X(t))\alpha_k(t)dt\right\}^2\right] + \int_Q E|h(t,X(t))|^2 dt \\
&\qquad - 2E\left[\sum_{k=1}^{n}\left\{\int_Q h(t,X(t))\alpha_k(t)dt\right\}^2\right]
\end{aligned}$$

$\to 0$ as $n \to \infty$ by Parseval's equation.

Therefore

$$\begin{aligned}
\int_Q h(t,X(t))dW(t) &= \underset{n\to\infty}{\mathrm{l.i.m.}}\sum_{k=1}^{n}\int_Q h(t,X(t))\alpha_k(t)dt\int_Q \alpha_k(s)dW(s) \\
&= \int_Q h(t,X(t))d^*W(t)
\end{aligned}$$

almost surely.

COROLLARY 1.1. *Under the same assumptions as in Theorem* 1, *if* B *is a Borel subset of* Q, *then*

$$\int_B h(t, X(t))d^*W(t) = \int_B h(t, X(t))dW(t)$$

almost surely.

COROLLARY 1.2. *If* h *is in* $L_2(Q)$, *then*

$$\int_Q h(t)d^*W(t) = \int_Q h(t)dW(t)$$

almost surely.

The following stochastic integration by parts formula [**13**, p. 268] (also stated in [**19**, p. 283] is well known.

THEOREM A. *If* $W(t)$ *and* $X(t)$ *are two independent standard Wiener processes on* $[0,T]$, *then for* $0 \le a < b \le T$,

$$\int_a^b X(t)d^*W(t) = X(b)W(b) - X(a)W(a) - \int_a^b W(t)d^*X(t)$$

almost surely.

In [**10**], Chang, Johnson, and Skoug obtained the following useful stochastic integration formulas.

THEOREM B. *If* $X(t)$ *and* $W(s,t)$ *are two independent standard Wiener processes on* $[0,T]$ *and* $Q = [0,T]^2$, *respectively, then*

$$\int_Q X(s)d^*W(s,t) = \int_0^T X(s)d^*W(s,T)$$

almost surely. Furthermore, if $h(s,t)$ *is of bounded variation on* Q, *then*

$$\int_Q h(s,t)X(s)d^*W(s,t) = \int_0^T \left[\int_{B(s)} h(u,v)dW(u,v)\right] d^*X(s)$$

almost surely where $B(s) \equiv [s,T] \times [0,T]$.

Our next result is a Fubini-type theorem which subsumes both theorems A and B. In particular note that we simply require the function g to be in L_2 rather than to be of bounded variation.

THEOREM 2. *Let* $W(t)$ *be a standard Wiener process on* $Q = [0,T]^N$ *and let* $Y(s)$ *be a standard Wiener process defined on* $Q' = [0,T]^k$. *Assume that* $W(t)$ *and* $Y(s)$ *are independent. Then for any function* g *in* $L_2(Q \times Q')$, *we have that*

$$\int_Q \left[\int_{Q'} g(s,t)dY(s)\right] dW(t) = \int_{Q'} \left[\int_Q g(s,t)dW(t)\right] dY(s)$$

almost surely.

Proof. It suffices to show that

$$E\left\{\int_Q \left[\int_{Q'} g(s,t)dY(s)\right] dW(t) - \int_{Q'} \left[\int_Q g(s,t)dW(t)\right] dY(s)\right\}^2 = 0. \tag{2.1}$$

Now,

$$E\left\{\int_Q\left[\int_{Q'} g(s,t)dY(s)\right]dW(t)\right\}^2 = E\left\{\int_Q\left[\int_{Q'} g(s,t)dY(s)\right]^2 dt\right\}$$
$$= \int_Q\left[\int_{Q'} g^2(s,t)ds\right]dt = \int_{Q'}\left[\int_Q g^2(s,t)dt\right]ds$$
$$= E\left\{\int_{Q'}\left[\int_Q g(s,t)dW(t)\right]dY(s)\right\}^2.$$

Also,

$$E\left\{\int_Q\left[\int_{Q'} g(s,t)dY(s)\right]dW(t)\cdot\int_{Q'}\left[\int_Q g(s,t)dW(t)\right]dY(s)\right\}$$
$$= E\left\{\int_Q\left[\int_{Q'} g(s,t)dY(s)\right]\cdot\int_{Q'}\left[\int_Q g(s,u)dW(u)\right]dY(s)\,dW(t)\right\}$$
$$= E\left\{\int_Q\left(\int_{Q'} g(s,t)\left[\int_Q g(s,u)dW(u)\right]ds\right)dW(t)\right\}$$
$$= \int_{Q'}\left[\int_Q g^2(s,t)dt\right]ds.$$

Thus, (2.1) is established.

The following corollaries and remarks show that Theorem 2 is indeed quite a general, powerful, and unifying theorem.

COROLLARY 2.1. *Let $W(t)$ and $Y(s)$ be as in Theorem 2 with $k \le N$. For $t = (t_1,\dots,t_N)$ in Q let $t^{(k)} = (t_1,\dots,t_k)$. Then for any function $h \in L_2(Q)$, we have*

$$\int_Q h(t)Y(t^{(k)})dW(t) = \int_{Q'}\left[\int_{B_k(s)} h(t)dW(t)\right]dY(s)$$

almost surely, where $B_k(s) = \prod_{j=1}^k [s_j,T]\times[0,T]^{N-k}$.

PROOF. Define a function ξ by

$$\xi(s;t^{(k)}) = \begin{cases} 1 & \text{if } 0 \le s_j \le t_j \le T \text{ for } j = 1,\dots,k, \\ 0 & \text{otherwise.}\end{cases} \tag{2.2}$$

Then

$$h(t)Y(t^{(k)}) = h(t)\int_0^{t_k}\cdots\int_0^{t_1} dY(s) = \int_{Q'} h(t)\xi(s;t^{(k)})dY(s).$$

The result now follows by setting $g(s,t) = h(t)\xi(s;t^{(k)})$ in Theorem 2 above.

COROLLARY 2.2. *Let $W(t), Y(s)$ and h be as in Corollary 2.1. Then*

$$\int_Q h(t)Y(t_{j_1},\dots,t_{j_k})dW(t) = \int_{Q'}\left[\int_{B_k^*(s)} h(t)dW(t)\right]dY(s_{j_1},\dots,s_{j_k})$$

almost surely where $j_1,\dots,j_k$ are k numbers in increasing order chosen from $\{1,2,\dots,N\}$ and $B_k^*(s)$ is obtained from $[0,T]^N$ by replacing all the j_ith factors by $[s_{j_i},T]$ for $i = 1,\dots,k$.

REMARK 2.1. It is easy to observe that Corollary 2.1 with $N = k = 1$ implies Theorem A above.

REMARK 2.2. It is also easy to observe the Corollary 2.1 with $N = 2$ and $k = 1$ implies Theorem B. Note that in Theorem B, h is required to be of bounded variation on Q, while in Corollary 2.1, h is only required to be in $L_2(Q)$.

REMARK 2.3. In [**28**, Theorem 2.1] we stated a special case of Corollary 2.2 which played a key role in the proof of the main result in [**28**].

REMARK 2.4. Theorem 2 also implies a stochastic integration (P.W.Z. and Itô) by parts formula where the integrator and integrand are independent standard Wiener processes on $Q = [0,T]^N$ for $N = 1$ (Theorem A above), $2, 3, \ldots$. We will give the proof for the case $N = 2$; the general case is similar but notationally more complicated. The stochastic integration by parts formula is the same as the ordinary Riemann-Stieltjes integration by parts formula [**33**, p. 415] but with each R-S integral $\int f dg$ replaced with the corresponding P.W.Z. integral $\int f d^* g$.

Let $R = [a,b] \times [\alpha,\beta] \subseteq Q = [0,T]^2$. Then, using Corollary 2.1, it follows that

$$\begin{aligned}
\int_a^b \int_\alpha^\beta Y(t_1,t_2) d^*W(t_1,t_2) &= \int_0^T \int_0^T \chi_R(t_1,t_2) Y(t_1,t_2) d^*W(t_1,t_2) \\
&= \int_0^T \int_0^T \left[\int_{s_1}^T \int_{s_2}^T \chi_R(t_1,t_2) dW(t_1,t_2) \right] d^*Y(s_1,s_2) \\
&= \int_0^a \int_0^\alpha \left[\int_a^b \int_\alpha^\beta dW(t_1,t_2) \right] d^*Y(s_1,s_2) \\
&\quad + \int_a^b \int_0^\alpha \left[\int_{s_1}^b \int_\alpha^\beta dW(t_1,t_2) \right] d^*Y(s_1,s_2) \\
&\quad + \int_0^a \int_\alpha^\beta \left[\int_a^b \int_{s_2}^\beta dW(t_1,t_2) \right] d^*Y(s_1,s_2) \\
&\quad + \int_a^b \int_\alpha^\beta \left[\int_{s_1}^b \int_{s_2}^\beta dW(t_1,t_2) \right] d^*Y(s_1,s_2)
\end{aligned}$$

almost surely. We next evaluate the inner integrals above, reduce the resulting two dimensional integrals whenever possible using the corollaries of Theorem 2, simplify, and obtain the stochastic parts formula

$$\begin{aligned}
&\int_a^b \int_\alpha^\beta Y(t_1,t_2) d^*W(t_1,t_2) \\
&\quad = W(b,\beta)Y(b,\beta) - W(b,\alpha)Y(b,\alpha) - W(a,\beta)Y(a,\beta) + W(a,\alpha)Y(a,\alpha) \\
&\qquad - \int_a^b [W(s_1,\beta) d^*Y(s_1,\beta) - W(s_1,\alpha) d^*Y(s_1,\alpha)] \\
&\qquad - \int_\alpha^\beta [W(b,s_2) d^*Y(b,s_2) - W(a,s_2) d^*Y(a,s_2)] \\
&\qquad + \int_a^b \int_\alpha^\beta W(s_1,s_2) d^*Y(s_1,s_2)
\end{aligned}$$

almost surely.

REMARK 2.5. The stochastic integration by parts formula for $N = 2$ was also established in [**21**] using a very different method.

3. Double stochastic integrals and centered double stochastic integrals. In §2 we established various Fubini-type theorems for stochastic integrals involving independent standard Wiener processes Y and W. In this section we consider the case where $Y = W$. Note that in this case, Theorem 1 no longer holds. For example, in the one-parameter case the P.W.Z. integral $\int_0^T W(t)d^*W(t)$ equals $W^2(T)/2$ almost surely, while the Itô integral $\int_0^T W(t)dW(t)$ equals $(W^2(T)-T)/2$ almost surely.

Let $\{\alpha_i(t)\}_{i=1}^{\infty}$ be a CON set on $Q = [0,T]^N$ with each α_i of bounded variation on Q. For g in $L_2(Q^2)$, we define the centered double stochastic integral of P.W.Z. by

$$\begin{aligned}(\mathrm{c})\int_{Q^2} g(s,t)\,d^*W(s)\,d^*W(t)\\ = \underset{n\to\infty}{\mathrm{l.i.m.}}\Bigg\{\Bigg[\sum_{j=1}^{n}\int_Q\Bigg[\sum_{i=1}^{n}\int_Q g(s,t)\alpha_i(s)ds\cdot\int_Q\alpha_i(u)dW(u)\Bigg]\alpha_j(t)dt\\ \cdot\int_Q\alpha_j(v)dW(v)\Bigg]\\ -\sum_{i=1}^{n}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_i(t)\,ds\,dt\Bigg\}.\end{aligned}$$

We may rewrite the above as

$$\begin{aligned}(\mathrm{c})\int_{Q^2} g(s,t)d^*W(s)d^*W(t)\\ = \underset{n\to\infty}{\mathrm{l.i.m.}}\Bigg[\sum_{i,j=1}^{n}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_j(t)\,ds\,dt\cdot\int_Q\alpha_i dW\cdot\int_Q\alpha_j dW\\ -\sum_{i=1}^{n}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_i(t)\,ds\,dt\Bigg].\end{aligned}$$

Although defined quite differently, the above centered double stochastic integral behaves very much like the one defined by Shepp [**31**, p. 333]. Since

$$E\left[\int_Q\alpha_i dW\cdot\int_0\alpha_j dW\right] = \delta_{ij},$$

we have that

$$E\left[(\mathrm{c})\int_{Q^2} g(s,t)d^*W(s)d^*W(t)\right] = 0.$$

Using the well-known formula (see [**30**, Problem 4E, p. 95]),

$$E\left[\int_Q\alpha_i dW\cdot\int_Q\alpha_j dW\cdot\int_Q\alpha_p dW\cdot\int_Q\alpha_q dW\right] = \delta_{ij}\cdot\delta_{pq}+\delta_{ip}\cdot\delta_{jq}+\delta_{iq}\cdot\delta_{jp},$$

we have that

$$E\left[\sum_{i,j=1}^{n}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_j(t)\,ds\,dt\cdot\int_Q \alpha_i dW \cdot\int_Q \alpha_j dW - \sum_{i=1}^{n}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_i(t)\,ds\,dt\right]^2$$

$$= \sum_{i,j=1}^{n}\left[\int_{Q^2} g(s,t)\alpha_i(s)\alpha_j(t)\,ds\,dt\right]^2 + \sum_{i,j=1}^{n}\left[\int_{Q^2} g(s,t)\alpha_i(s)\alpha_j(t)\,ds\,dt\right]\left[\int_{Q^2} g(s,t)\alpha_j(s)\alpha_i(t)\,ds\,dt\right].$$

Due to Parseval's equation, we obtain

$$E\left[(\mathrm{c})\int_{Q^2} g(s,t)d^*W(s)d^*W(t)\right]^2 = \int_{Q^2} g^2(s,t)\,ds\,dt + \int_{Q^2} g(s,t)g(t,s)\,ds\,dt.$$

As is well known (see [**16**]), Itô's double stochastic integral $\int_{Q^2} g(s,t)dW(s)dW(t)$ has expected value 0 and variance $2\int_{Q^2} g^2(s,t)\,ds\,dt$. So,

$$(\mathrm{c})\int_{Q^2} g(s,t)d^*W(s)d^*W(t)$$

and $\int_{Q^2} g(s,t)dW(s)dW(t)$ have different variances unless $g(s,t)$ is symmetric in s and t.

Let $\mathscr{A}$ consist of all functions $g \in L_2(Q^2)$ such that

$$\sum_{i=1}^{\infty}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_i(t)\,ds\,dt$$

converges for some CON set $\{\alpha_i(t)\}$ on Q with each $\alpha_i(t)$ of bounded variation. Park [**27**] shows that for a large class of functions $g \in L_2(Q^2)$,

$$\sum_{i=1}^{\infty}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_i(t)\,ds\,dt = \int_Q g(s,s)ds.$$

In particular, this class contains all the continuous functions on Q^2.

Let $g \in \mathscr{A}$. Define the double stochastic integral of P.W.Z. by

$$\int_{Q^2} g(s,t)d^*W(s)d^*W(t) = \underset{n\to\infty}{\mathrm{l.i.m.}}\sum_{i,j=1}^{n}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_j(t)\,ds\,dt\cdot\int_Q \alpha_i dW\cdot\int_Q \alpha_j dW.$$

It follows that

$$\int_{Q^2} g(s,t)d^*W(s)d^*W(t) = (\mathrm{c})\int_{Q^2} g(s,t)d^*W(s)d^*W(t) + \sum_{i=1}^{\infty}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_i(t)\,ds\,dt.$$

Therefore

$$E\left[\int_{Q^2} g(s,t)d^*W(s)d^*W(t)\right] = \sum_{i=1}^{\infty}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_i(t)\,ds\,dt,$$

and

$$\operatorname{Var}\left[\int_{Q^2} g(s,t)d^*W(s)d^*W(t)\right] = \int_{Q^2} g^2(s,t)\,ds\,dt + \int_{Q^2} g(s,t)g(t,s)\,ds\,dt.$$

For $g \in \mathscr{A}$ define the iterated stochastic integral by the formula

$$\int_Q \left[\int_Q g(s,t)dW(s)\right] d^*W(t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{i=1}^{n}\int_Q \left[\int_Q g(s,t)dW(s)\right]\alpha_i(t)dt \cdot \int_Q \alpha_i dW.$$

We have the following.

THEOREM 3. *If $g \in \mathscr{A}$, then, almost surely,*

$$\int_{Q^2} g(s,t)d^*W(s)d^*W(t) = \int_Q \left[\int_Q g(s,t)dW(s)\right] d^*W(t).$$

PROOF. The theorem follows by observing that

$$E\left\{\sum_{i,j=1}^{n}\int_{Q^2} g(s,t)\alpha_i(s)\alpha_j(t)\,ds\,dt \cdot \int_Q \alpha_i dW \cdot \int_Q \alpha_j dW \right.$$
$$\left. - \sum_{k=1}^{m}\int_Q \left[\int_Q g(s,t)dW(s)\right]\alpha_k(t)dt \cdot \int_Q \alpha_k dW\right\}^2 \to 0$$

as $m, n \to \infty$.

COROLLARY 3.1. *If $g \in \mathscr{A}$, then, almost surely,*

$$\int_Q \left[\int_Q g(s,t)dW(s)\right] d^*W(t) = \int_Q \left[\int_Q g(s,t)dW(t)\right] d^*W(s).$$

PROOF. By the definition,

$$\int_{Q^2} g(s,t)d^*W(s)d^*W(t) = \int_{Q^2} g(s,t)d^*W(t)d^*W(s).$$

Therefore, the result follows from Theorem 3.

THEOREM 4. *Let $h \in L_2(Q)$. For $t = (t_1,\ldots,t_N) \in Q$ and $k = 1,2,\ldots,N$, let $t^{(k)} = (t_1,\ldots,t_k)$, $t_T^{(k)} = (t_1,\ldots,t_k,T,\ldots,T)$ and $\tilde{t}^{(k)} = (t_{k+1},\ldots,t_N)$. Then for $k = 1,2,\ldots,N$,*

$$\int_Q h(t)W(t_T^{(k)})d^*W(t) = \int_Q \left[\int_{B_k(s)} h(t)dW(t)\right] d^*W(s) \tag{3.1}$$

almost surely where $B_k(s) = \prod_{j=1}^{k}[s_j,T] \times [0,T]^{N-k}$.

PROOF. Using the function ξ given by (2.2), we may write

$$h(t)W(t_T^{(k)}) = \int_Q h(t)\xi(s^{(k)};t^{(k)})dW(s)$$

almost surely. Then using Corollary 3.1 we obtain

$$\int_Q h(t)W(t_T^{(k)})d^*W(t) = \int_Q \left[\int_Q h(t)\xi(s^{(k)};t^{(k)})dW(s)\right] d^*W(t)$$
$$= \int_Q \left[\int_Q h(t)\xi(s^{(k)};t^{(k)})dW(t)\right] d^*W(s).$$

Thus, the result follows from Theorem 3, provided the function $h(t)\xi(s^{(k)};t^{(k)}) \in \mathscr{A}$. Now, choose a CON set $\{\beta_p(s^{(k)})\}_{p=1}^{\infty}$ on $[0,T]^k$ satisfying:

$$\underset{n\to\infty}{\text{l.i.m.}} \sum_{i=1}^{n} \beta_p(t^{(k)}) \int_0^{t_k} \cdots \int_0^{t_1} \beta_p(s^{(k)})ds_1 \cdots ds_k = 2^{-k}.$$

For the existence of such a CON set, see Park [**25**, pp. 391–92]. Also, choose a CON set $\{\gamma_q(\tilde{s}^{(k)})\}_{q=1}^{\infty}$ on $[0,T]^{N-k}$, and set $\{\alpha_i(s)\} = \{\beta_p(s^{(k)})\gamma_q(\tilde{s}^{(k)})\}$. Then,

$$\sum_{i=1}^{\infty} \int_{Q^2} h(t)\xi(s^{(k)};t^{(k)})\alpha_i(s)\alpha_i(t)\,ds\,dt$$
$$= \sum_{p,q=1}^{\infty} \int_Q h(t) \left[\beta_p(t^{(k)}) \int_0^{t_k} \cdots \int_0^{t_1} \beta_p(s^{(k)})ds_1 \cdots ds_k\right]$$
$$\cdot \left[\gamma_q(\tilde{t}^{(k)}) \int_{[0,T]^{N-k}} \gamma_q(\tilde{s}^{(k)})d\tilde{s}^{(k)}\right] dt$$
$$= 2^{-k} \int_Q h(t)dt.$$

Therefore, $h(t)\xi(s^{(k)};t^{(k)}) \in \mathscr{A}$.

Our final corollary is a Fubini-type theorem similar to Corollary 2.1.

COROLLARY 4.1. *Let $h \in L_2(Q)$. Then, for $k = 1, 2, \ldots, N$,*

$$\int_Q h(t)W(t_T^{(k)})d^*W(t) = \int_{[0,T]^k} \left[\int_{B_k(s)} h(t)dW(t)\right] d^*W(s_T^{(k)})$$

almost surely, where the right-hand side is a P.W.Z. stochastic integral with respect to a CON set on $[0,T]^k$.

PROOF. Because of Theorem 4, it suffices to show that

$$\int_Q \left[\int_{B_k(s)} h(t)dW(t)\right] d^*W(s) = \int_{[0,T]^k} \left[\int_{B_k(s)} h(t)dW(t)\right] d^*W(s_T^{(k)})$$

almost surely. However, this follows almost immediately by observing that

$$\int_{[0,T]^k} \alpha(s^{(k)})dW(s_T^{(k)}) = \int_Q \alpha(s^{(k)})dW(s).$$

REMARK. One cannot establish a theorem like Theorem 4 in terms of Itô's stochastic integral, because $W(t_T^{(k)})$ is not measurable with respect to the σ-field generated by $\{W(s)\colon s_i \le t_i, 1 \le i \le N\}$. Thus, the left-hand side of (3.1) is not defined in the sense of Itô.

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