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THE INFLUENCE OF THE INITIAL DISTRIBUTION ON A RANDOM WALK

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ABSTRACT. Let $T_1, T_2, \ldots$ be i.i.d. random variables, $S_n = T_1 + \cdots + T_n$; let X and Y be independent of $(T_n)_{n \geq 1}$. We study the total variation distance between the distributions of $X + S_n$ and $Y + S_n$, especially its speed of convergence to 0 in the case that some S_j is not singular.

1. Introduction. Let $T_1, T_2, \ldots$ be a sequence of independent, identically distributed random variables and $S_n := T_1 + \cdots + T_n$. If T_1 is not concentrated on some lattice, $E(T_1) = 0$ and $\sigma^2 = E(T_1^2) < \infty$,

$$\sigma(2\pi n)^{1/2} P(S_n \in I) \to \lambda(I), \quad \text{as } n \to \infty, \tag{1.1}$$

for all intervals $I \subset \mathbf{R}$, where λ denotes the Lebesgue measure (Shepp (**1964**), Stone (**1965**, **1967**); see also Breiman (**1968**, Chapter 10). In this sense S_n is asymptotically "uniformly distributed on $\mathbf{R}$".

Let X and Y be random variables which are independent of $(T_n)_{n \geq 1}$. In this note we study the distance between the distributions of $X + S_n$ and $Y + S_n$, measured by means of the total variation metric

$$\|P^{X+S_n} - P^{Y+S_n}\| = \sup_B |P(X + S_n \in B) - P(Y + S_n \in B)|. \tag{1.2}$$

Here the supremum is taken over all Borel subsets B of $\mathbf{R}$, and, for a random variable U, P^U denotes its distribution. Especially in renewal theory so-called delayed renewal processes $(X + S_n)_{n \geq 1}$ are often considered. It is intuitively clear that the initial term X can have no great influence on the asymptotic behavior of $(X + S_n)_{n \geq 1}$. This idea has e.g. been used in Lindvall's (**1977**) proof of the renewal theorem. We shall show that

$$d_n^X := \|P^{X+S_n} - P^{S_n}\| \leq (aE|X| + b)n^{-1/2}, \tag{1.3}$$

if some P^{S_j} has an absolutely continuous component. We shall also see what happens in the case when all P^{S_j} are singular. The constants $a, b \geq 0$ in (1.3) depend only on P^{T_1}. Clearly

$$\|P^{X+S_n} - P^{Y+S_n}\| \leq d_n^X + d_n^Y, \tag{1.4}$$

so that (1.3) also gives an estimate for (1.2).

No assumptions about moments are needed for (1.3). We note however that if $E(T_1^2) < \infty$, we also have

$$d_n^X \geq cn^{-1/2} \tag{1.5}$$

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for some $c > 0$, if $X \equiv x = \text{const.} \neq 0$. (1.5) follows from Chebyshev's inequality which gives

$$P(|S_n - nE(T_1)| < \sigma n^{1/2}) \geq (n-1)/n, \tag{1.6}$$

where $\sigma^2 = \operatorname{Var}(T_1)$. Let $x > 0$; the case $x < 0$ is treated similarly. (1.6) implies that there is a subinterval $I = (t, t+x]$ of $(nE(T_1) - \sigma n^{1/2}, nE(T_1) + \sigma n^{1/2}]$ satisfying

$$P(S_n \in I) \geq \frac{n-1}{n}\left(\left[\frac{2\sigma n^{1/2}}{x}\right] + 1\right)^{-1} \geq cn^{-1/2}. \tag{1.7}$$

Therefore,

$$\|P^{X+S_n} - P^{S_n}\| \geq P(S_n \leq t + x) - P(S_n \leq t) \geq cn^{-1/2}.$$

Hence if $E(T_1^2) < \infty$, d_n^X is of the same order as $n^{-1/2}$.

On the other hand, if T_1 has the symmetric stable distribution with characteristic function $\exp(-|\varsigma|^\alpha)$, $0 < \alpha \leq 2$, the following limiting relation holds for $X \equiv x \neq 0$:

$$\lim_{n\to\infty} n^{1/\alpha} d_n^X = \frac{|x|}{\pi\alpha}\Gamma\left(\frac{1}{\alpha}\right). \tag{1.8}$$

Thus d_n^X can tend to 0 much faster than $n^{-1/2}$.

To see (1.8), note that the density f_α of T_1 is symmetric around 0 and unimodal and that S_n has the density $n^{-1/\alpha} f_\alpha(n^{-1/\alpha} t)$. Hence if $x > 0$,

$$\begin{aligned} n^{1/\alpha}\|P^{x+S_n} - P^{S_n}\| &= \frac{1}{2}\int_{-\infty}^{\infty} |f_\alpha(n^{-1/\alpha}t) - f_\alpha(n^{-1/\alpha}(t-x))|\,dt \\ &= \int_{-\infty}^{x/2} f_\alpha(n^{-1/\alpha}t)\,dt - \int_{-\infty}^{-x/2} f_\alpha(n^{-1/\alpha}t)dt \\ &= 2n^{1/\alpha}\int_0^{x/2n^{1/\alpha}} f_\alpha(s)ds \to x f_\alpha(0) \\ &= \frac{x}{\pi}\int_0^\infty e^{-\varsigma^\alpha}d\varsigma = \frac{x}{\pi\alpha}\Gamma\left(\frac{1}{\alpha}\right), \end{aligned} \tag{1.9}$$

where for the fourth equation Fourier inversion is used. For $x < 0$, (1.8) follows by symmetry.

II. The proof. Some further notation is necessary. $\mu * \nu$ denotes the convolution of the signed measures μ and ν, and $\mu^{*i} := \mu * \cdots * \mu$ (i factors). ε_x is the point mass at $x \in \mathbf{R}$.

THEOREM. *Assume that $P^{S_j} = \alpha Q_1 + (1-\alpha)Q_2$ for some $j \in \mathbf{N}$ and $\alpha \in (0,1]$, where Q_1 and Q_2 are probability measures and $Q_1 \ll \lambda$. Then there are constants $a, b \geq 0$ depending only on P^{T_1} such that*

$$\|P^{X+S_n} - P^{S_n}\| \leq (aE|X| + b)n^{-1/2}. \tag{2.1}$$

PROOF. Let $q := dQ_1/d\lambda$. By decomposing Q_1 into $Q_1 1_{\{q\leq N\}}$ and $Q_1 1_{\{q>N\}}$ we see that Q_1 can be assumed to have a bounded Lebesgue density. Then $Q_1 * Q_1$ has a continuous Lebesgue density which of course must exceed some $\varepsilon > 0$ on some interval I. Therefore $Q_1 * Q_1$ can be written as $\beta R + (1-\beta)Q'$, where R denotes the rectangular distribution on I, $\beta > 0$ and Q' is some probability measure. Hence we can assume that

$$P^{S_j} = \gamma R + (1-\gamma)Q \tag{2.2}$$

for some $j \in \mathbf{N}$, $\gamma \in (0,1]$ and some probability measure Q. Since

$$\begin{aligned} \|P^{X+S_n} - P^X\| &\le \sup_B \int |P(x+S_n \in B) - P(S_n \in B)|\, dP^X(x) \\ &\le \int \|P^{x+S_n} - P^{S_n}\|\, dP^X(x), \end{aligned} \tag{2.3}$$

it suffices to prove that for some $a, b \ge 0$

$$\|P^{x+S_n} - P^{S_n}\| \le (a|x| + b)n^{-1/2} \quad \text{for all } x \in \mathbf{R}. \tag{2.4}$$

Let $n = ij + k$, where $k \in \{0, 1, \ldots, j-1\}$ and j satisfies (2.2). Then by (2.2),

$$\begin{aligned} \|P^{x+S_n} - P^{S_n}\| &= \|[(\gamma R + (1-\gamma)Q)^{*i} * \varepsilon_x - (\gamma R + (1-\gamma)Q)^{*i}] * P^{S_k}\| \\ &\le \|(\gamma R + (1-\gamma)Q)^{*i} * (\varepsilon_x - \varepsilon_0)\| \\ &= \left\| \sum_{l=0}^{i} \binom{i}{l} \gamma^l (1-\gamma)^{i-l} R^{*l} Q^{*(i-l)} * (\varepsilon_x - \varepsilon_0) \right\| \\ &\le \sum_{l=0}^{i} \binom{i}{l} \gamma^l (1-\gamma)^{i-l} \|R^{*l} * (\varepsilon_x - \varepsilon_0)\|. \end{aligned} \tag{2.5}$$

We have twice used the inequality $\|\mu * \nu\| \le \|\mu\|$ which is valid for arbitrary signed measures μ and probability measures ν. Now we split the sum at the right-hand side of (2.5) into two terms: the sums over all l for which $|l - \gamma i| > \gamma i^{3/4}$ or $|l - \gamma i| \le \gamma i^{3/4}$, respectively. The first sum is, by Chebyshev's inequality, not larger than

$$\begin{aligned} \sum_{\substack{0 \le l \le i \\ |l-\gamma i| > \gamma i^{3/4}}} \binom{i}{l} \gamma^l (1-\gamma)^{i-l} &\le \frac{\gamma(1-\gamma)i}{(\gamma i^{3/4})^2} \\ &= \gamma^{-1}(1-\gamma) i^{-1/2} \le K n^{-1/2} \end{aligned} \tag{2.6}$$

for some constant K, because i is the integer part of n/j and thus of the same order as n. The other sum can be estimated from above by

$$\sup\left\{ \|P^{x+\widetilde{S}_l} - P^{\widetilde{S}_l}\| \,\middle|\, \gamma(i - i^{3/4}) \le l \le i \right\}, \tag{2.7}$$

where $\widetilde{S}_1 := U_1 + \cdots + U_l$ for i.i.d. random variables $U_1, U_2, \ldots$ *distributed according to* R. Therefore the assertion will be proved, if we can show that

$$\|P^{x+\widetilde{S}_l} - P^{\widetilde{S}_l}\| \le (a|x| + b)l^{-1/2}. \tag{2.8}$$

The density f_l of $\widetilde{S}_l$ is given in Feller (**1971**), p. 28. It is unimodal and symmetric around lu, if u is the center of the interval I. It is clear that one can, without loss of generality, assume that $u = 0$.

Now it is easily seen that if $x > 0$,

$$\|P^{x+\widetilde{S}_l} - P^{\widetilde{S}_l}\| = \frac{1}{2}\int_{-\infty}^{\infty} |f_l(t) - f_l(t-x)|\,dt$$

$$(2.9) \qquad = \int_{-\infty}^{x/2} (f_l(t) - f_l(t-x))\,dt$$

$$= \int_{-(x/2)}^{x/2} f_l(t)dt = P(|\widetilde{S}_l| \le x/2).$$

Let Φ and φ be the distribution function and the density of $N(0,1)$. Then $\Phi(y) \le \frac{1}{2} + \varphi(0)y = \frac{1}{2} + (2\pi)^{-1/2}y$ for all $y > 0$. By the Berry-Esseen inequality (Feller (**1971**), p. 542) we obtain

$$(2.10) \qquad P\left(|\widetilde{S}_l| \le \frac{x}{2}\right) \le 2\Phi\left(\frac{x}{2\sqrt{l\,\mathrm{Var}(U_1)}}\right) - 1 + 2\frac{3E|U_1|^3}{l^{1/2}[\mathrm{Var}(U_1)]^{3/2}}$$

$$\le l^{-1/2}[(2\pi\mathrm{Var}(U_1))^{-1/2}x + 6E|U_1|^3[\mathrm{Var}(U_1)]^{-3/2}].$$

The case $x < 0$ is treated similarly. (2.8) is proved.

Finally we consider the case when all P^{S_n} are singular with respect to Lebesgue measure. Assume that

$$(2.11) \qquad \sum_{n=1}^{\infty} P^{S_n} \perp \lambda.$$

Then we assert that

$$(2.12) \qquad P^{x+S_n} \perp P^{S_n} \quad \text{for all } n \in \mathbf{N}$$

holds for Lebesgue—almost all $x \in \mathbf{R}$. For suppose on the contrary that

$$(2.13) \qquad \|P^{x+S_n} - P^{S_n}\| < 1$$

for some $n \in \mathbf{N}$ and all $x \in A_n$, where $A_n \subset \mathbf{R}$ satisfies $\lambda(A_n) > 0$. Then there is an $N \in \mathbf{N}$ such that $0 < \lambda(B_n) < \infty$ for $B_n := A_n \cap [-N, N]$. Define the probability measure P_n on $\mathbf{R}$ by

$$(2.14) \qquad P_n(B) := \lambda(B_n)^{-1} \int_{B_n} P^{x+S_n}(B)\,dx.$$

Then $P_n \ll \lambda$ and

$$(2.15) \qquad \|P_n - P^{S_n}\| \le \lambda(B_n)^{-1} \int_{B_n} \|P^{x+S_n} - P^{S_n}\|\,dx < 1.$$

Thus P^{S_n} is not singular with respect to P_n and, consequently, has an absolutely continuous component.

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