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LARGE AUTOMORPHISM GROUPS OF HYPERELLIPTIC KLEIN SURFACES

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ABSTRACT. In this paper we study the groups of automorphisms of all hyperelliptic bordered Klein surfaces of genus $p > 3$ having at least $4(p-1)$ automorphisms.

1. Introduction. We mean as Klein surface a compact surface with a dianalytic structure (see [**1**]). An automorphism of the surface X is a dianalytic homeomorphism $f\colon X \to X$.

A bordered surface X is called hyperelliptic if it admits an automorphism φ of order two such that the quotient X/φ has algebraic genus zero (see [**4**, p. 144]).

Bordered Klein surfaces of algebraic genus p have at most $12(p-1)$ automorphisms [**9**, Theorem 5]. In this paper we study the groups of automorphisms of all hyperelliptic surfaces of genus $p > 3$ having at least $4(p-1)$ automorphisms. In case $p = 2$ or $p = 3$, all possible automorphism groups have been obtained in [**6** and **5**]. In particular, for $p = 2$, D_4 and D_6 are cases of order > 4: for $p = 3$, D_6 and $Z_2 \times D_4$ are the groups of order > 8.

2. Preliminaries. The main tool in our study is the theory of non-Euclidean crystallographic (NEC) groups. An NEC group is a discrete subgroup of the isometries group of the hyperbolic plane, including orientation-reversing elements, with compact quotient space.

NEC groups are classified by their signature, that is

$$(*) \qquad (g, \pm, [m_1, \dots, m_r], \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}).$$

The m_i's are the proper periods, the $(n_{i1}, \dots, n_{is_i})$ are the period-cycles and the n_{ij} the periods of the period-cycles. The signature determines a presentation of the group, given by generators

(i) x_i, $i = 1, \dots, r$,
(ii) e_i, $i = 1, \dots, k$,
(iii) c_{ij}, $i = 1, \dots, k$, $j = 0, \dots, s_i$,
(iv) (if sign '+') a_i, b_i, $i = 1, \dots, g$,
(if sign ' − ') d_i, $i = 1, \dots, g$,

and relations

(i) $x_i^{m_i} = 1$, $i = 1, \dots, r$,

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(ii) $e_i^{-1}c_{i0}e_ic_{is_i} = 1$, $i = 1, \ldots, k$,
(iii) $c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1}c_{ij})^{n_{ij}} = 1$, $i = 1, \ldots, k$, $j = 1, \ldots, s_i$,
(iv) (if sign '+') $x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1$,
(if sign '−') $x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2 = 1$.

An NEC group Γ with signature $(*)$ has an associated area

$$|\Gamma| = 2\pi\left(\alpha g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right),$$

$\alpha = 1$ if the sign is '−' and 2 if the sign is '+'.

A Klein surface X of topological genus g with k boundary components may be expressed as D/Γ, D being the hyperbolic plane and Γ an NEC group with signature $(g, \pm, [-], \{(-), \overset{k}{\ldots}, (-)\})$, the sign being '+' if X is orientable, and '−' if nonorientable. The relation between algebraic and topological genera, and number of boundary components, is $p = \alpha g + k - 1$.

If G is a group of automorphisms of D/Γ, it may be expressed as Γ'/Γ, where Γ' is an NEC group of which Γ is a normal subgroup. Then the order of G, $|G|$, is $|\Gamma|/|\Gamma'|$. Hence, if $|G| \geq 4(p-1)$, one obtains by a simple calculation, keeping in mind that if Γ' has a nonempty period-cycle, this has at least two consecutive periods equal to two [**7**, §4] that Γ' has one of the following signatures:

(i)	$(0, +, [-], \{(2, 2, 2, n)\})$,	$\lvert G\rvert = \frac{4n}{n-2}(p-1)$,
(ii)	$(0, +, [-], \{(2, 2, 3, 3)\})$,	
(iii)	$(0, +, [3], \{(2, 2)\})$,	$\lvert G\rvert = 6(p-1)$,
(iv)	$(0, +, [2, 3], \{(-)\})$,	
(v)	$(0, +, [-], \{(2, 2, 3, 4)\})$,	$\lvert G\rvert = \frac{24}{5}(p-1)$,
(vi)	$(0, +, [-], \{(2, 2, 3, 5)\})$,	$\lvert G\rvert = \frac{30}{7}(p-1)$,
(vii)	$(0, +, [-], \{(2, 2, 2, 2, 2)\})$,	
(viii)	$(0, +, [2], \{(2, 2, 2)\})$,	
(ix)	$(0, +, [-], \{(2, 2, 4, 4)\})$,	$\lvert G\rvert = 4(p-1)$,
(x)	$(0, +, [4], \{(2, 2)\})$,	
(xi)	$(0, +, [2, 4], \{(-)\})$,	
(xii)	$(0, +, [-], \{(2, 2, 3, 6)\})$,	

(This list, for groups of order $\geq 4p$, may be deduced from [**9**].)

3. Large automorphism groups. In [**4**, Theorem 4.5] hyperelliptic Klein surfaces have been characterized in the following way.

Let Γ be an NEC group with signature $(g, \pm, [-], \{(-), \overset{k}{\ldots}, (-)\})$. Then D/Γ is hyperelliptic if and only if there exists a unique NEC group Γ_1 with $|\Gamma_1 : \Gamma| = 2$, and whose signature is

(1) $(0, +, [-], \{(2, \overset{2k}{\ldots}, 2)\})$ if $g = 0$,
(2) $(0, +, [2, \overset{2g+k}{\ldots}, 2], \{(-)\})$ if $g \neq 0$ and Γ has sign '+' (and then $k < 3$),
(3) $(0, +, [2, \overset{g}{\ldots}, 2], \{(2, \overset{2k}{\ldots}, 2)\})$ if Γ has sign '−'.

When $G = \Gamma'/\Gamma$ is the automorphism group of the hyperelliptic surface D/Γ then $\Gamma_1 \triangleleft \Gamma'$. We call $G_1 = \Gamma'/\Gamma_1$, and it is cyclic or dihedral, since it is the automorphism group of the disc D/Γ_1.

In this section we obtain all hyperelliptic surfaces of genus $p > 3$ with at least $4(p-1)$ automorphisms. It was proven in [**3**] that no such surface has $12(p-1)$ automorphisms.

Now we state three technical lemmas.

LEMMA 1. *If* Γ' *has signature* (ii), (iii) *or* (iv), *and there exists a normal subgroup* Γ *of* Γ' *having signature* $(g, \pm, [-], \{(-), \overset{k}{\ldots}, (-)\})$, D/Γ *being a hyperelliptic surface of algebraic genus p, such that* Γ'/Γ *is the automorphism group of* D/Γ *having order* $6(p-1)$, *then* $p \leq 2$.

PROOF. Let Γ' be a group with signature (ii). If there exists Γ in the above conditions, there is an epimorphism $\theta_1 : \Gamma' \to G_1$ such that $\ker \theta_1 = \Gamma_1$, the group of the hyperellipticity of D/Γ. If Γ_1 had signature (1), θ_1 should be defined by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(c_{13}) = y$, $\theta_1(c_{14}) = x$, which is impossible because $c_{12}c_{13}$ has order three: if Γ_1 had signature (2) or (3) we may not obtain proper periods equal to two in $\ker \theta_1$. Hence θ_1 exists in no case.

If Γ' has signature (iii), and Γ_1 has signature (1), then $\theta_1(c_{10}) = 1$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(e_1) = x$, $\theta_1(x_1) = x^{-1}$; as $|G_1| = 3(p-1)$, we have $3(p-1) = 3$, and so $p = 2$. If Γ_1 has signature (2) we may not obtain simultaneously proper periods and period-cycles, and so θ_1 does not exist, and the same holds in case Γ_1 has signature (3).

If Γ' has signature (iv), Γ_1 may not have signature (1) or (3) because it has no nonempty period-cycles. If Γ_1 has signature (2), then $\theta_1(x_1) = 1$, $\theta_1(x_2) = x$, $\theta_1(e_1) = x^{-1}$, $\theta_1(c_{10}) = 1$. Then $3(p-1) = 3$, and so $p = 2$.

LEMMA 2. *If* Γ' *has signature* (v) (*respectively* (vi)), *there exists no normal subgroup* Γ *of* Γ' *having signature* $(g, \pm, [-], \{(-), \overset{k}{\ldots}, (-)\})$, D/Γ *being a hyperelliptic surface of algebraic genus p, such that* Γ'/Γ *is the automorphism group of* D/Γ *having order* $\frac{24}{5}(p-1)$ (*respectively* $\frac{30}{7}(p-1)$).

PROOF. If there exists Γ in the above conditions we may define an epimorphism $\theta_1 : \Gamma' \to G_1$, G_1 having order $\frac{12}{5}(p-1)$ (respectively $\frac{15}{7}(p-1)$) with kernel Γ_1 having signature (1), (2) or (3).

If Γ_1 has signature (1), θ_1 is given by $\theta_1(c_{10}) = 1$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = x$, $\theta_1(c_{13}) = y$, $\theta_1(c_{14}) = 1$. This possibility holds only for case (v), and then G_1 is generated by x, y, satisfying $x^2 = y^2 = (xy)^3 = 1$. Hence $|G_1| = 6 = \frac{12}{5}(p-1)$, and p is not an integer.

When Γ_1 has signature (2), the epimorphism θ_1 is defined by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = y$, $\theta_1(c_{13}) = z$, $\theta_1(c_{14}) = x$, where xz has order 4 (respectively 5), yz has order 3, and so $y \neq 1$. Then the number of proper periods of $\ker \theta_1, p+1$, equals $\frac{12}{5}(p-1)/4$ (respectively $\frac{15}{7}(p-1)/5$) [**2**, Theorem 2]. Hence $\frac{3}{5}(p-1) = p+1$ (respectively $\frac{3}{7}(p-1) = p+1$), impossible.

Finally, one may not get an epimorphism θ_1 with kernel having signature (3).

LEMMA 3. *If* Γ' *has signature* (vii), (viii), (ix), (x), (xi) *or* (xii) *and there exists a normal subgroup* Γ *of* Γ' *having signature* $(g, \pm, [-], \{(-), \overset{k}{\ldots}, (-)\})$, D/Γ *being a hyperelliptic surface of algebraic genus p, such that* Γ'/Γ *is the automorphism group of* D/Γ *having order* $4(p-1)$, *then* $p \leq 3$.

PROOF. Let Γ' have signature (vii). Then we have an epimorphism θ_1 from Γ' onto G_1 with kernel Γ_1. If Γ_1 has signature (1), θ_1 is defined by $\theta_1(c_{10}) = x$,

$\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(c_{13}) = y$, $\theta_1(c_{14}) = z$, $\theta_1(c_{15}) = x$; if $y = z = 1$, then $G_1 = Z_2$, and so $2(p-1) = 2$, $p = 2$; if $y = 1$, $z \neq 1$, then we call r the order of xz and the number of period-cycles of ker $\theta_1, 1$, equals $2(p-1)/2r$ [**7**, §3]; so $r = p-1$; and the number of periods of the period-cycle, $2k$, equals $4r$ [**7**], and so $2r = k$. Hence $2(p-1) = k$ and $p = k/2+1$; since in case (1) $p = k-1$, we obtain $p = 3$. Finally, if $y \neq 1 \neq z$, calling again r the order of xz, we obtain $r = p-1$, and the number of periods of the period-cycle, $2k$, is now $2r$. Hence $2k = 2r$, and so $p+1 = p-1$, impossible. For any other election of y and z, the kernel of θ_1 would not have signature (1).

If Γ_1 has signature (2), the epimorphism θ_1 is given by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = y$, $\theta_1(c_{13}) = z$, $\theta_1(c_{14}) = t$, $\theta_1(c_{15}) = x$. Then the number of proper periods of ker θ_1 is $\beta(p-1)$, that must equal $p+1$, and so $p = 2$ or $p = 3$.

When Γ_1 has signature (3), θ_1 is defined by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(c_{13}) = y$, $\theta_1(c_{14}) = z$, $\theta_1(c_{15}) = x$; if $y = 1$, $z = x$, then the number of period-cycles of $\ker\theta_1$, that is 1, equals $2(p-1)/2$; so $p = 2$; if $y = z = x$, it results again $p = 2$; if $z = y \neq x$, then calling r the order of xy, the number of period-cycles, 1, is $2(p-1)/2r$, and so $r = p-1$, and the number of periods in the period-cycle, $2k$, is $2r$; so $r = k$. Since the number of proper periods, g, is $2(p-1)/2$, we have $p = g+1$, $p = k+1$, $p = g+k-1$, and so $p = 3$. If y and z have not the considered values, the kernel of θ_1 has not signature (3).

Let now Γ' have signature (viii). If Γ_1 has signature (1), the epimorphism θ_1 is given by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(c_{13}) = y$, $\theta_1(x_1) = z$, $\theta_1(e_1) = z^{-1}$. Then, if $y = z = 1$, $G_1 = Z_2$, and so $p = 2$. If $y \neq 1 \neq z$, let r be the order of yz. Then the number of period-cycles of the kernel, 1, is $2(p-1)/2r$, and $r = p-1$; and the number of periods of the period-cycle, $2k$, is $2r$. So $r = k$, and $p = k+1$, impossible.

If Γ_1 has signature (2), the epimorphism θ_1 satisfies $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = y$, $\theta_1(c_{13}) = z$, $\theta_1(x_1) = t$, $\theta_1(e_1) = t^{-1}$. The number of proper periods, $p+1$, is $\beta(p-1)$, and so $p = 2$, or 3.

If Γ_1 has signature (3), then θ_1 is given by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(c_{13}) = y$, $\theta_1(x_1) = 1$, $\theta_1(e_1) = 1$. If $y = z = 1$, $G_1 = Z_2$ and so $p = 2$; if $y \neq 1 \neq z$, let r be the order of yz, and then the number of period-cycles, 1, is $2(p-1)/2r$, and $r = p-1$; the number of periods of the period-cycle, $2k$, is $2r$, so $r = k$; and the number of proper periods, g, is $2(p-1)$. Hence $p = 2$.

When Γ' has signature (ix) and Γ_1 has signature (1), θ_1 is given by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(c_{13}) = y$, $\theta_1(c_{14}) = x$, xy having order 4. Then $G_1 = D_4$, and so $2(p-1) = 8$; so $p = 5$, $k = 6$. Thus there exists $\theta\colon \Gamma' \to G$, $|G| = 16$, and ker θ has signature $(0, +, [-], \{(-), (-), (-), (-), (-), (-)\})$. As then 6 would have to divide 16 [**7**], this is impossible.

If Γ_1 has signature (2), θ_1 is given by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = y$, $\theta_1(c_{13}) = z$, $\theta_1(c_{14}) = x$. Then the number of proper periods, $p+1$, is $2(p-1)/4$ or $2(2(p-1)/4)$ [**2**].

If Γ_1 has signature (3), the epimorphism is $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(c_{13}) = y$, $\theta_1(c_{14}) = x$. Since xy has order two, $1 = 2(p-1)/4$, and so $p = 3$.

If Γ' has signature (x), when Γ_1 has signature (1), then $\theta_1(c_{10}) = 1$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(x_1) = x$, $\theta_1(e_1) = x^{-1}$, with $x^4 = 1$. Hence $G_1 = Z_4$, and $p = 3$.

When Γ_1 has signature (2), then $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = y$, $\theta_1(x_1) = z$, $\theta_1(e_1) = z^{-1}$, with $z^2 = 1$. Then the number of proper periods, $p+1$, is $2(p-1)/2$, impossible.

Γ_1 having signature (3), we have $\theta_1(c_{10}) = 1$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(x_1) = x$, $\theta_1(e_1) = x^{-1}$. Then $G_1 = Z_2$, and $p = 2$.

When Γ' has signature (xi), Γ_1 must have signature (2) because it may not have nonempty period-cycles. In this case θ_1 is defined by $\theta_1(x_1) = x$, $\theta_1(x_2) = y$, $\theta_1(e_1) = y^{-1}x^{-1}$, $\theta_1(c_{10}) = 1$. The number of proper periods, $p+1$, is $\beta(p-1)$, $1 \leq \beta \leq 3$. Hence $p = 2$ or 3.

Finally, Γ' having signature (xii), one applies the same arguments of the last case of Lemma 2.

Thus the unique signature that we must consider is (i), $(0, +, [-], \{(2,2,2,n)\})$. Besides, $n > 3$, since there are no surfaces of genus $p > 3$ with $12(p-1)$ automorphisms.

In next theorems we obtain the group G of automorphisms of D/Γ we are looking for. Since $G = \Gamma'/\Gamma$ and the surface is hyperelliptic, there exists Γ_1 such that $\Gamma \triangleleft \Gamma_1 \triangleleft \Gamma'$. Hence there exist epimorphisms $\theta_1 \colon \Gamma' \to Z_2$ with $\ker \theta_1 = G_1$ and $\theta \colon \Gamma' \to Z_2$ with $\ker \theta = G$ and $|G_1| = \frac{1}{2}|G|$. The technique we use starts constructing θ_1 and so determining conditions for G_1: and from G_1 and the construction of θ, conditions for G are obtained. Finally the possible groups G are checked.

THEOREM 1. *Let D/Γ be a hyperelliptic Klein surface of algebraic genus $p > 3$, where Γ has signature $(0, +, [-], \{(-), \overset{k}{\ldots}, (-)\})$. Then, if $G = \Gamma'/\Gamma$ is the automorphism group of D/Γ, having order at least $4(p-1)$, $G = D_{p+1} \times Z_2$, and Γ' has signature $(0, +, [-], \{(2,2,2,p+1)\})$. Moreover, for each p, there exists a hyperelliptic Klein surface D/Γ in the above conditions, having $D_{p+1} \times Z_2$ as its automorphism group.*

PROOF. If Γ_1 is the group of hyperellipticity of D/Γ, Γ_1 has signature $(0, +, [-], \{(2, \overset{2k}{\ldots}, 2)\})$. If $G = \Gamma'/\Gamma$ is the automorphism group of D/Γ, having order $\frac{4n}{n-2}(p-1)$, there is an epimorphism $\theta_1 \colon \Gamma' \to G_1$, having kernel Γ_1, and $|G_1| = \frac{2n}{n-2}(p-1)$. The epimorphism θ_1 is defined by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(c_{13}) = y$, $\theta_1(c_{14}) = x$. Since Γ_1 has no proper periods, the order of xy is n. As Γ_1 has 1 period-cycle, we have $\frac{2n}{n-2}(p-1)/2n = 1$, and so $n = p+1$. So G_1 is generated by x, y, satisfying $x^2 = y^2 = (xy)^{p+1} = 1$, and $G_1 = D_{p+1}$.

Now we have an epimorphism θ from Γ' onto G having kernel Γ. If α is a central element of G and π is the natural epimorphism from G onto $G/\langle\alpha\rangle = G_1$, then θ_1 must be equal to $\pi\theta$, since D/Γ is a hyperelliptic surface.

As $|G| = 4(p+1)$, then θ is defined by $\theta(c_{10}) = x$ or $x\alpha$, $\theta(c_{11}) = 1$ or α, $\theta(c_{12}) = 1$ or α, $\theta(c_{13}) = y$ or $y\alpha$, $\theta(c_{14}) = x$ or $x\alpha$, where $\theta(c_{10}) = \theta(c_{14})$ and $\theta(c_{11}) \neq \theta(c_{12})$; and $x^2 = y^2 = (x\langle\alpha\rangle y\langle\alpha\rangle)^{p+1} = 1$. The group G is generated by x, y, α, and so $G = D_{p+1} \times Z_2$ or $D_{2(p+1)}$.

If $G = D_{p+1} \times Z_2$, for each p, let Γ' have signature $(0, +, [-], \{(2,2,2,p+1)\})$ and θ be the epimorphism from Γ' onto

$$D_{p+1} \times Z_2 = \langle x, y, \alpha \,|\, x^2 = y^2 = \alpha^2 = (xy)^{p+1} = 1, \alpha \text{ central}\rangle,$$

given by $\theta(c_{10}) = x$, $\theta(c_{11}) = \alpha$, $\theta(c_{12}) = 1$, $\theta(c_{13}) = y$, $\theta(c_{14}) = x$. The kernel of θ is orientable [**8**], and the signature of $\Gamma = \ker\theta$ is $(0, +, [-], \{(-), \overset{k}{\ldots}, (-)\})$ [**3, 7**].

This group $D_{p+1} \times Z_2$ is the full automorphism group of the surface, as it admits no other group of higher order.

If $G = D_{2(p+1)}$, there exists no epimorphism θ from Γ' onto G since Γ' has no element of order $2(p+1)$.

Now we study the surfaces D/Γ in which Γ has signature $(g, +, [-], \{(-), \overset{k}{\ldots}, (-)\})$.

THEOREM 2. *Let D/Γ be a hyperelliptic Klein surface of algebraic genus $p > 3$, where Γ has signature $(g, +, [-], \{(-), \overset{k}{\ldots}, (-)\})$, $g > 0$, (and then $k = 1$ or 2). Let $G = \Gamma'/\Gamma$ be the automorphism group of D/Γ, and suppose that G has order at least $4(p-1)$. Then $G = D_{p+1} \times Z_2$ (only for p odd) or D_{2p} (for every p) and Γ' has signature, respectively, $(0, +, [-], \{(2,2,2,p+1)\})$ and $(0, +, [-], \{(2,2,2,2p)\})$. Moreover, for each p, there exist hyperelliptic Klein surfaces D/Γ in each of the above conditions.*

PROOF. Let Γ_1 be the group of the hyperellipticity of D/Γ. Then Γ_1 has signature $(0, +, [2, \overset{2g+k}{\ldots}, 2], \{(-)\})$, and there exists an epimorphism θ_1 from Γ' onto G_1 with kernel Γ_1. This epimorphism is defined by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = y$, $\theta_1(c_{13}) = z$, $\theta_1(c_{14}) = x$.

(a) If xz has order n, since Γ_1 has proper periods, $y = z$. As Γ_1 has 1 period-cycle, $\frac{2n}{n-2}(p-1)/2n = 1$, and so $n = p+1$. Thus G_1 is generated by x, y, satisfying $x^2 = y^2 = (xy)^{p+1} = 1$, and $G_1 = D_{p+1}$.

The epimorphism θ from Γ' onto G must again satisfy $\theta_1 = \pi\theta$, π being the epimorphism from G onto $G/\langle\alpha\rangle$. The epimorphism θ is so defined by $\theta(c_{10}) = x$ or $x\alpha$, $\theta(c_{11}) = 1$, $\theta(c_{12}) = y$ or $y\alpha$, $\theta(c_{13}) = y$ or $y\alpha$, $\theta(c_{14}) = x$ or $x\alpha$, where $\theta(c_{10}) = \theta(c_{14})$, $\theta(c_{12}) \neq \theta(c_{13})$, and $x^2 = y^2 = (x\langle\alpha\rangle y\langle\alpha\rangle)^{p+1} = 1$. Thus G is generated by x, y, α, and $G = D_{p+1} \times Z_2$ or $D_{2(p+1)}$.

Let $p+1$ be even. Then, if $G = D_{p+1} \times Z_2$, we take Γ' having signature $(0, +, [-], \{(2,2,2,p+1)\})$. The epimorphism θ from Γ' onto

$$D_{p+1} \times Z_2 = \langle x, y, \alpha \mid x^2 = y^2 = \alpha^2 = (xy)^{p+1} = 1, \alpha \text{ central}\rangle$$

is given by $\theta(c_{10}) = x$, $\theta(c_{11}) = 1$, $\theta(c_{12}) = y\alpha$, $\theta(c_{13}) = y$, $\theta(c_{14}) = x$. The kernel of θ is orientable [8] and the signature of $\Gamma = \ker\theta$ is $(\frac{p-1}{2}, +, [-], \{(-), (-)\})$.

If $G = D_{2(p+1)}$, there exists no epimorphism θ as Γ' has no elements of order $2(p+1)$.

In the valid case, $p+1$ even, $D_{p+1} \times Z_2$ is the full automorphism group of the surface.

(b) If xz has order $n/2$ and $y = z$, since Γ_1 has 1 period-cycle we have $\frac{2n}{n-2}(p-1)n = 1$, and so $p = n/2$. Hence G_1 is generated by x, y, satisfying $x^2 = y^2 = (xy)^p = 1$, and $G_1 = D_p$.

The epimorphism θ from Γ' onto G must be defined by $\theta(c_{10}) = x$ or $x\alpha$, $\theta(c_{11}) = 1$, $\theta(c_{12}) = y$ or $y\alpha$, $\theta(c_{13}) = y$ or $y\alpha$, $\theta(c_{14}) = x$ or $x\alpha$, where $\theta(c_{10}) = \theta(c_{14})$, $\theta(c_{12}) \neq \theta(c_{13})$, and $x^2 = y^2 = (x\langle\alpha\rangle y\langle\alpha\rangle)^p = 1$. Thus G is generated by x, y, α, and G is $D_p \times Z_2$ or D_{2p}.

When $G = D_p \times Z_2$, for each p, let Γ' have signature $(0, +, [-], \{(2,2,2,2p)\})$ and θ be the epimorphism from Γ' onto $D_p \times Z_2$. If p is even, the group $D_p \times Z_2$ has no elements of order $2p$, and so θ does not exist.

If $G = D_{2p} = \langle x', y' \mid x'^2 = y'^2 = (x'y')^{2p} = 1\rangle$, and θ is given by $\theta(c_{10}) = x'$, $\theta(c_{11}) = 1$, $\theta(c_{12}) = y'(x'y')^p$, $\theta(c_{13}) = y'$, $\theta(c_{14}) = x'$, the kernel of θ is orientable

[8] and $\Gamma = \ker\theta$ has signature $(\frac{p-1}{2}, +, [-], \{(-), (-)\})$ (p odd) or $(\frac{p}{2}, +, [-], \{(-)\})$ (p even). The group D_{2p} is the full automorphism group of the surface.

(c) If xz has order $n/2$ and $y \neq z$, the number of proper periods of Γ_1 is $p+1 = \frac{2n}{n-2}(p-1)/n = \frac{2}{n-2}(p-1)$, and so $p = n/(4-n)$, impossible since $n \geq 1$.

THEOREM 3. *Let D/Γ be a hyperelliptic Klein surface of algebraic genus $p > 3$, where Γ has signature $(g, -, [-], \{(-), \overset{k}{\ldots}, (-)\})$. Then if $G = \Gamma'/\Gamma$ is the automorphism group of D/Γ, having order at least $4(p-1)$, we have $g = 1$ (and so $k = p$) and $G = D_{2p}$, Γ' having signature $(0, +, [-], \{(2,2,2,2p)\})$. Moreover, for each p, there exists a hyperelliptic Klein surface D/Γ in the above conditions.*

PROOF. Let Γ_1 be the group of the hyperellipticity of D/Γ. Then Γ_1 has signature $(0, +, [2, \overset{g}{\ldots}, 2], \{(2, \overset{2k}{\ldots}, 2)\})$ and there exists θ_1 epimorphism from Γ' onto G_1 with kernel Γ_1. This epimorphism is defined by $\theta_1(c_{10}) = x$, $\theta_1(c_{11}) = 1$, $\theta_1(c_{12}) = 1$, $\theta_1(c_{13}) = y$, $\theta_1(c_{14}) = x$. In this situation either $x = y$ or xy has order $n/2$. If $x = y$, the group $G_1 = Z_2$ and $p < 2$, impossible. Hence $x \neq y$, $(xy)^{n/2} = 1$. Since Γ_1 has 1 period-cycle, $1 = \frac{2n}{n-2}(p-1)/n$ and so $p = n/2$. Since the period-cycle has $2k$ periods, it is $2k = n$, and $k = n/2$.

G_1 is so generated by x, y, satisfying $x^2 = y^2 = (xy)^p = 1$, and $G_1 = D_p$. The epimorphism θ from Γ' onto G with kernel Γ satisfies $\theta_1 = \pi\theta$ where π is the epimorphism from G onto $G/\langle\alpha\rangle$. Hence θ is defined by $\theta(c_{10}) = x$ or $x\alpha$, $\theta(c_{11}) = 1$ or α, $\theta(c_{12}) = 1$ or α, $\theta(c_{13}) = y$ or $y\alpha$, $\theta(c_{14}) = x$ or $x\alpha$, with $\theta(c_{10}) = \theta(c_{14})$, $\theta(c_{11}) \neq \theta(c_{12})$, $x^2 = y^2 = (x\langle\alpha\rangle y\langle\alpha\rangle)^p = 1$. Thus G is generated by x, y, α, and $G = D_p \times Z_2$ or D_{2p}.

When $G = D_p \times Z_2$, for each p, let Γ' have signature $(0, +, [-], \{(2,2,2,2p)\})$ and θ be the epimorphism from Γ' onto $D_p \times Z_2$. If p is even, G has not elements of order $2p$, and θ does not exist.

If $G = D_{2p} = \langle x', y' \mid x'^2 = y'^2 = (x'y')^{2p} = 1\rangle$, θ is given by $\theta(c_{10}) = x'$, $\theta(c_{11}) = 1$, $\theta(c_{12}) = (x'y')^p$, $\theta(c_{13}) = y'$, $\theta(c_{14}) = x'$. The kernel of θ is nonorientable [8] and its signature is $(1, -, [-], \{(-), \overset{p}{\ldots}, (-)\})$. The group D_{2p} is the full group of automorphisms of the surface.

These results may be summarized in the following table:

	Hyperelliptic surface D/Γ Signature of Γ $(p > 3)$	Signature of Γ'	Group $G = \Gamma'/\Gamma$
(i)	$(0, +, [-], \{(-), \overset{p-1}{\ldots}, (-)\})$	$(0, +, [-], \{(2,2,2,p+1)\})$	$D_{p+1} \times Z_2$
(ii)	$(\frac{p}{2}, +, [-], \{(-)\})$	$(0, +, [-], \{(2,2,2,2p)\})$	D_{2p}
(iii)	$(\frac{p-1}{2}, +, [-], \{(-), (-)\})$	$(0, +, [-], \{(2,2,2,p+1)\})$ $(0, +, [-], \{(2,2,2,2p)\})$	$D_{p+1} \times Z_2$ D_{2p}
(iv)	$(1, -, [-], \{(-), \overset{p}{\ldots}, (-)\})$	$(0, +, [-], \{(2,2,2,2p)\})$	D_{2p}

Observe that nonorientable hyperelliptic Klein surfaces of genus $p > 3$ and topological genus $g \geq 2$ have always less than $4(p-1)$ automorphisms.

REMARKS. (1) Observe that May obtains in [**10**, pp. 275, 279] examples of Klein surfaces with the topological type and the automorphism group that appear in cases (i) and (iv) of the above table.

(2) Also in [**10**, Theorems 2,3] it was proven that infinite values of p verify that the order of the automorphism group of an orientable (nonorientable) surface of

genus p is at most $4(p+1)(4p)$. Observe that if the surface is hyperelliptic, this upper bound is obtained for all values of p.

Observe too that the table gives the upper bound for the order of the automorphism group of a hyperelliptic surface, according to its topological type.

(3) May proved in [**11**] that the upper bound of the order of an automorphism of a Klein surface of genus p is $2(p+1)$ for orientable surfaces of even p, and $2p$ otherwise. We show in the above table families of hyperelliptic surfaces attaining these bounds, coinciding in some cases with the examples of [**11**].

4. Real algebraic curves. All results on bordered Klein surfaces can be stated in terms of real algebraic curves, since given a Klein surface of algebraic genus p with k boundary components, there exists a real projective irreducible curve of genus p with k connected components, whose group of birational isomorphisms coincides with the automorphism group of the surface, and conversely. The surface is orientable if and only if the curve disconnects its complexification.

Consequently, this equivalence allows us to determine all hyperelliptic real curves of genus $p > 3$ with at least $4(p-1)$ birational isomorphisms, by a simple translation of the above results.

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