A CONVERGENCE PROBLEM CONNECTED WITH CONTINUED FRACTIONS

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(Communicated by Larry J. Goldstein)

ABSTRACT. The set $Z_{\alpha} := \{\beta \mid \lim_{n \to \infty} \|q_n \beta\| = 0\}$ is considered, where $(q_n)_{n \in \mathbb{N}}$ is the sequence of best approximation denominators of $\alpha$, and it is explicitly determined for $\alpha$ with bounded continued fraction coefficients.

Introduction. Let the irrational number $\alpha$ have continued fraction expansion $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ and let $q_{-1} = 0$, $1 = q_0 < q_1 < q_2 < \cdots$ with $q_{i+1} = a_{i+1} q_i + q_{i-1}$ be the best approximation denominators of $\alpha$. By Theorem 4.3 in [2] it follows, that $\{q_n \beta\}_{n \in \mathbb{N}}$ (where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of $x$) is uniformly distributed modulo one for almost all $\beta \in \mathbb{R}$ in the sense of Lebesgue measure. Clearly the sequence is not uniformly distributed for all $\beta$, since if we take $\beta = m\alpha + n$ with $m, n \in \mathbb{Z}$, then $\lim_{n \to \infty} \|q_n \beta\| = 0$ where $\|x\|$ denotes the distance from $x$ to the nearest integer.

We ask now for the set $Z_{\alpha} := \{\beta \in \mathbb{R} \mid \lim_{n \to \infty} \|q_n \beta\| = 0\}$. The problem of determining this set also is of some importance in some problems of automata theory [3]. We will show

THEOREM 1. If $\alpha$ has bounded continued fraction coefficients, then

\[ \lim_{n \to \infty} \|q_n \beta\| = 0 \]

if and only if $\beta = m\alpha + n$ with $m, n \in \mathbb{Z}$.

This in general is not true if $\alpha$ does not have bounded continued fraction coefficients, for we can show

THEOREM 2. There are $\alpha \in \mathbb{R}$ and $\beta \neq m\alpha + n$ for all $m, n \in \mathbb{Z}$ with

\[ \lim_{n \to \infty} \|q_n \beta\| = 0. \]

PROOFS. For the proof of Theorem 1 we need two lemmata. In these two lemmata let $\alpha$ be a fixed real number and the $q_i$ as above.

LEMMA 1. Let $a_i \leq K - 1$ for all $i \in \mathbb{N}$ and let $n, p_n, p_{n+1} \in \mathbb{N}$ be given, then there exists at most one $\beta \in [0, 1)$ with

\[ |q_n \beta - p_n| \leq \frac{1}{4K}, \quad |q_{n+1} \beta - p_{n+1}| \leq \frac{1}{4K} \]

and

\[ \|q_m \beta\| \leq \frac{1}{4K} \quad \text{for all } m \geq n. \]

Received by the editors October 14, 1986 and, in revised form, June 23, 1987.

1980 Mathematics Subject Classification (1985 Revision). Primary 10A32, 10F20.

Key words and phrases. Continued fractions.
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PROOF. For any $\beta$ which fulfills our conditions, we have
\[ q_{n+2} \beta \in I_{n+2} := \left( p_{n+1} \cdot \frac{q_{n+2}}{q_{n+1}} - \frac{q_{n+2}/q_{n+1} + q_{n+2}/q_{n+1}}{4K}, p_{n+1} \cdot \frac{q_{n+2}}{q_{n+1}} + \frac{q_{n+2}/q_{n+1}}{4K} \right). \]

The length of $I_{n+2}$ is $\leq \frac{1}{2}$ and because of $\|q_{n+2} \beta\| \leq 1/4K$, there is, independent of $\beta$, exactly one $p_{n+2}$ with $|q_{n+2} \beta - p_{n+2}| \leq 1/4K$. Going on this way, we see, that independent of $\beta$, by $n, p_n$ and $p_{n+1}$ all further values $p_{n+2}, p_{n+3}, \ldots$ are determined, for which $|q_n \beta - p_n| \leq 1/4K$. Therefore $\beta$ is unique, because $\beta = \lim_{m \to \infty} (p_m/q_m)$.

LEMMA 2. Let $\beta, p_n$ and $p_{n+1}$ be such that $|q_n \beta - p_n| < 1/8K, |q_{n+1} \beta - p_{n+1}| < 1/8K, p_n/q_n \neq p_{n+1}/q_{n+1}$, $\text{sgn}(q_n \beta - p_n) = -\text{sgn}(q_{n+1} \beta - p_{n+1})$ and $\|q_m \beta\| < 1/4K$ for $m \geq n$; if we define $\lambda_n := [0; a_n, a_{n+1}, \ldots]$ and $\alpha_0 := (p_{n+1} + \lambda_{n+1} p_n)/(q_{n+1} + \lambda_{n+1} q_n)$, then $\beta = \alpha_0$.

PROOF. We show $|q_n \alpha_0 - p_n| < 1/4K, |q_{n+1} \alpha_0 - p_{n+1}| < 1/4K$ and $\|q_m \alpha_0\| < 1/4K$ for $m \geq n$ and then by Lemma 1 the result follows. We have
\[ |q_n \cdot p_{n+1} + \lambda_{n+1} p_n - p_n| = \left| \frac{q_n p_{n+1} - q_{n+1} p_n}{q_{n+1} + \lambda_{n+1} q_n} \right| \leq \left| \frac{q_n p_{n+1} - q_{n+1} p_n}{q_{n+1} + \lambda_{n+1} q_n} \right| \leq 1/4K \]
and
\[ \left| q_{n+1} \cdot p_{n+1} + \lambda_{n+1} p_n - p_{n+1} \right| = \left| \lambda_{n+1} \cdot \frac{p_{n+1} q_{n+1} - q_{n+1} p_{n+1}}{q_{n+1} + \lambda_{n+1} q_n} \right| \leq 1/4K. \]

Now we consider in the plane the lattice $\Gamma$ produced by the vectors $(q_n, p_n)$ and $(q_{n+1}, p_{n+1})$ which are independent, and the line
\[ g: \quad y = \frac{p_{n+1} + \lambda_{n+1} p_n}{q_{n+1} + \lambda_{n+1} q_n} \cdot x. \]

This lattice is isomorphic to the lattice $\Gamma'$ produced by $(0,1), (1,0)$ and the line $g': \ y = \lambda_{n+1} \cdot x$.

Now we consider the Klein-model of the development of $\lambda_{n+1}$ to continued fractions as it is explained for example in [1]. $\Gamma'$ is the lattice for this development.

By observing the same operations in $\Gamma$ as in $\Gamma'$ according to this development we see that
\[ \|q_m \alpha_0\| \leq \max(\|q_n \alpha_0\|, \|q_{n+1} \alpha_0\|) < \frac{1}{4K} \quad \text{for all } m \geq n. \]
(See Figure 1 and Figure 2.)

For example we have (see Figure 3) $\|q_{n+3} \alpha_0\| = |R_2 T_2|$ and since $|P_2 S_2| \leq \max(|P_{-1} 0|, |P_0 S_0|)$ we have $|R_2 S_2| \leq \max(|R_{-1} 0|, |R_0 S_0|)$ and therefore
\[ |R_2 T_2| \leq \max(|R_{-1} T_0|, |R_0 T_1|) = \max(\|q_n \alpha_0\|, \|q_{n+1} \alpha_0\|), \]
and the result follows.

PROOF OF THEOREM 1. Let $\lim_n \|q_n \beta\| = 0$. Of course $\beta$ must be irrational or an integer. So let $\beta$ be irrational. We define by $p_n$ the integer lying next to $q_n \beta$, and $N_0$ to be such, that $\|q_n \beta\| < 1/8K$ for $n \geq N_0$. 

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We show, that there is a $n \geq N_0$ such, that $\text{sgn}(q_n\beta - p_n) = -\text{sgn}(q_{n+1}\beta - p_{n+1})$. Since $\beta \notin \mathbb{Q}$, from this it follows, that $p_n/q_n \neq p_{n+1}/q_{n+1}$.

If for example $q_n\beta - p_n > 0$ for all $n \geq N_0$, then for all $n \geq N_0 + 1$ we would have $q_{n+1}\beta = a_n q_n\beta + q_{n-1}\beta$. Further $q_n\beta = p_n + \varepsilon_0$, $q_{n-1}\beta = p_{n-1} + \varepsilon_1$ with $0 < \varepsilon_1 < 1/8K$ and therefore $q_{n+1}\beta = a_n p_n + p_{n-1} + a_n \varepsilon_0 + \varepsilon_1$ with $0 < a_n \varepsilon_0 + \varepsilon_1 < 1/8 + 1/8K \leq 1/4$ and therefore $\|q_{n+1}\beta\| = a_n \varepsilon_0 + \varepsilon_1 \geq \|q_n\beta\|$ for all $n \geq N_0 + 1$ and this is a contradiction since $\|q_{N_0+1}\beta\| \neq 0$. So by Lemma 2

$$\beta = \frac{p_{n+1} + \lambda_{n+1} p_n}{q_{n+1} + \lambda_{n+1} q_n}.$$  

If we define $Q_i$ such, that $|q_i \alpha - Q_i| \leq 1/q_{i+1}$ for all $i$, then (see [4]).

$$\lambda_{n+1} = [0; a_{n+1}, a_{n+2}, \ldots] = -\frac{Q_{n+1} - \alpha q_{n+1}}{Q_n - \alpha q_n}$$

and therefore

$$\beta = \frac{(p_{n+1} Q_n - Q_{n+1} p_n) + \alpha(q_{n+1} p_n - q_n p_{n+1})}{(q_{n+1} Q_n - Q_{n+1} q_n)} = \alpha a + b \quad \text{with } a, b \in \mathbb{Z}$$

because $|q_{n+1} Q_n - Q_{n+1} q_n| = 1$, and the theorem is proved.

**Proof of Theorem 2.** Let $\gamma$ and $\delta$ irrational be such that $\gamma = (a\gamma + b)/(c\gamma + d)$ for all $a, b, c, d \in \mathbb{Z}$ and such that $1, \gamma, \delta$ are linear independent over $\mathbb{Q}$. Therefore not both are of the form $e\alpha + f$ with $e, f \in \mathbb{Z}$. We define now a real number $\alpha$ by its partial quotients $a_0, a_1, \ldots$ such that for the best approximation denominators $q_n$ of $\alpha$ we have $\lim_{n \to \infty} \|q_n \gamma\| = 0$ and $\lim_{n \to \infty} \|q_n \delta\| = 0$.

Let $a_0 = 0$, $q_{-1} = 0$, $q_0 = 1$ and assume that $a_1, a_2, \ldots, a_n$ and therefore $q_1, q_2, \ldots, q_n$ are defined.
$1, q_n\gamma$ and $q_n\delta$ are linearly independent over $\mathbb{Q}$ and therefore the sequence $\{(kq_n\gamma), (kq_n\delta)\}_{k\in\mathbb{N}}$ is dense in the unit square. Therefore we can choose an integer $a_{n+1}$ such that

$$\max(\|a_{n+1}q_n\gamma + q_{n-1}\gamma\|, \|a_{n+1}q_n\delta + q_{n-1}\delta\|) \leq 1/n.$$ 

Then with $q_{n+1} = a_{n+1}q_n + q_{n-1}$ we have $\lim_{n \to \infty} \|q_n\beta\| = 0$ for $\beta = \gamma$ and for $\beta = \delta$ and the theorem is proved.
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