A CONVERGENCE PROBLEM CONNECTED WITH CONTINUED FRACTIONS

GERHARD LARCHER

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ABSTRACT. The set $Z_\alpha := \{ \beta \mid \lim_{n \to \infty} \| q_n \beta \| = 0 \}$ is considered, where $(q_n)_{n \in \mathbb{N}}$ is the sequence of best approximation denominators of $\alpha$, and it is explicitly determined for $\alpha$ with bounded continued fraction coefficients.

Introduction. Let the irrational number $\alpha$ have continued fraction expansion $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ and let $q_{-1} = 0, 1 = q_0 < q_1 < q_2 < \cdots$ with $q_{i+1} = a_{i+1} q_i + q_{i-1}$ be the best approximation denominators of $\alpha$. By Theorem 4.3 in [2] it follows, that $\{ q_n \beta \}_{n \in \mathbb{N}}$ (where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of $x$) is uniformly distributed modulo one for almost all $\beta \in \mathbb{R}$ in the sense of Lebesgue measure. Clearly the sequence is not uniformly distributed for all $\beta$, since if we take $\beta = m\alpha + n$ with $m, n \in \mathbb{Z}$, then $\lim_{n \to \infty} \| q_n \beta \| = 0$ where $\|x\|$ denotes the distance from $x$ to the nearest integer.

We ask now for the set $Z_\alpha := \{ \beta \in \mathbb{R} \mid \lim_{n \to \infty} \| q_n \beta \| = 0 \}$. The problem of determining this set also is of some importance in some problems of automata theory [3]. We will show

THEOREM 1. If $\alpha$ has bounded continued fraction coefficients, then

$$\lim_{n \to \infty} \| q_n \beta \| = 0$$

if and only if $\beta = m\alpha + n$ with $m, n \in \mathbb{Z}$.

This in general is not true if $\alpha$ does not have bounded continued fraction coefficients, for we can show

THEOREM 2. There are $\alpha \in \mathbb{R}$ and $\beta \neq m\alpha + n$ for all $m, n \in \mathbb{Z}$ with

$$\lim_{n \to \infty} \| q_n \beta \| = 0.$$ 

PROOFS. For the proof of Theorem 1 we need two lemmata. In these two lemmata let $\alpha$ be a fixed real number and the $q_i$ as above.

LEMMA 1. Let $a_i \leq K - 1$ for all $i \in \mathbb{N}$ and let $n, p_n, p_{n+1} \in \mathbb{N}$ be given, then there exists at most one $\beta \in [0, 1)$ with

$$|q_n \beta - p_n| \leq \frac{1}{4K}, \quad |q_{n+1} \beta - p_{n+1}| \leq \frac{1}{4K}$$

and

$$\| q_m \beta \| \leq \frac{1}{4K}$$

for all $m \geq n$.

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PROOF. For any $\beta$ which fulfills our conditions, we have

$$q_{n+2}\beta \in I_{n+2} := \left( p_{n+1} \cdot \frac{q_{n+2}}{q_{n+1}} - \frac{q_{n+2}/q_{n+1}}{4K}, p_{n+1} \cdot \frac{q_{n+2}}{q_{n+1}} + \frac{q_{n+2}/q_{n+1}}{4K} \right).$$

The length of $I_{n+2}$ is $\frac{1}{2}$ and because of $\|q_{n+2}\beta\| \leq \frac{1}{4K}$, there is, independent of $\beta$, exactly one $p_{n+2}$ with $|q_{n+2}\beta - p_{n+2}| \leq \frac{1}{4K}$. Going on this way, we see, that independent of $\beta$, by $n, p_n$ and $p_{n+1}$ all further values $p_{n+2}, p_{n+3}, \ldots$ are determined, for which $|q_m\beta - p_m| \leq \frac{1}{4K}$. Therefore $\beta$ is unique, because $\beta = \lim_{m \to \infty} (p_m/q_m).

LEMMA 2. Let $\beta, p_n$ and $p_{n+1}$ be such that $|q_n\beta - p_n| < \frac{1}{8K}, |q_{n+1}\beta - p_{n+1}| < \frac{1}{8K}, p_n/q_n \neq p_{n+1}/q_{n+1}$, $\text{sgn}(q_n\beta - p_n) = -\text{sgn}(q_{n+1}\beta - p_{n+1})$ and $\|q_m\beta\| < \frac{1}{4K}$ for $m \geq n$; if we define $\lambda_n := [0; a_n, a_{n+1}, \ldots]$ and $\alpha_0 := (p_{n+1} + \lambda_{n+1} q_n)/(q_{n+1} + \lambda_{n+1} q_n)$, then $\beta = \alpha_0$.

PROOF. We show $|q_n\alpha_0 - p_n| < \frac{1}{4K}, |q_{n+1}\alpha_0 - p_{n+1}| < \frac{1}{4K}$ and $\|q_m\alpha_0\| < \frac{1}{4K}$ for $m \geq n$ and then by Lemma 1 the result follows. We have

$$\left| q_n \cdot \frac{p_{n+1} + \lambda_{n+1} p_n}{q_{n+1} + \lambda_{n+1} q_n} - p_n \right| = \left| \frac{q_n p_{n+1} - q_{n+1} p_n}{q_{n+1} + \lambda_{n+1} q_n} \right| \leq \left| q_n \cdot \frac{p_{n+1} - p_n}{q_{n+1}} \right| \leq \frac{1}{4K}$$

and

$$\left| q_{n+1} \cdot \frac{p_{n+1} + \lambda_{n+1} p_n}{q_{n+1} + \lambda_{n+1} q_n} - p_{n+1} \right| = \left| \lambda_{n+1} \cdot \frac{p_n q_{n+1} - q_n p_{n+1}}{q_{n+1} + \lambda_{n+1} q_n} \right| \leq \frac{1}{4K}.$$

Now we consider in the plane the lattice $\Gamma$ produced by the vectors $(q_n, p_n)$ and $(q_{n+1}, p_{n+1})$ which are independent, and the line

$$g: \ y = \frac{p_{n+1} + \lambda_{n+1} p_n}{q_{n+1} + \lambda_{n+1} q_n} \cdot x.$$ 

This lattice is isomorphic to the lattice $\Gamma'$ produced by $(0,1), (1,0)$ and the line $g': \ y = \lambda_{n+1} \cdot x$.

Now we consider the Klein-model of the development of $\lambda_{n+1}$ to continued fractions as it is explained for example in [1]. $\Gamma'$ is the lattice for this development.

By observing the same operations in $\Gamma$ as in $\Gamma'$ according to this development we see that

$$\|q_m\alpha_0\| \leq \max(\|q_n\alpha_0\|, \|q_{n+1}\alpha_0\|) < \frac{1}{4K} \quad \text{for all } m \geq n.$$

(See Figure 1 and Figure 2.)

For example we have (see Figure 3) $\|q_{n+3}\alpha_0\| = |R_2T_2|$ and since $|P_2S_2| \leq \max(|P_{-1}0|, |P_0S_0|)$, we have $|R_2S_2| \leq \max(|R_{-1}0|, |R_0S_0|)$ and therefore

$$|R_2T_2| \leq \max(|R_{-1}T_0|, |R_0T_1|) = \max(\|q_n\alpha_0\|, \|q_{n+1}\alpha_0\|),$$

and the result follows.

PROOF OF THEOREM 1. Let $\lim_n \|q_n\beta\| = 0$. Of course $\beta$ must be irrational or an integer. So let $\beta$ be irrational. We define by $p_n$ the integer lying next to $q_n\beta$, and $N_0$ to be such, that $\|q_n\beta\| < \frac{1}{8K}$ for $n \geq N_0$. 

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We show, that there is a $n \geq N_0$ such, that $\text{sgn}(q_n \beta - p_n) = -\text{sgn}(q_{n+1} \beta - p_{n+1})$. Since $\beta \notin \mathbb{Q}$, from this it follows, that $p_n/q_n \neq p_{n+1}/q_{n+1}$.

If for example $q_n \beta - p_n > 0$ for all $n \geq N_0$, then for all $n \geq N_0 + 1$ we would have $q_{n+1} \beta = a_n q_n \beta + q_{n-1} \beta$. Further $q_n \beta = p_n + \varepsilon_0$, $q_{n-1} \beta = p_{n-1} + \varepsilon_1$ with $0 < \varepsilon_1 < 1/8 K$ and therefore $q_{n+1} \beta = a_n p_n + p_{n-1} + a_n \varepsilon_0 + \varepsilon_1$ with $0 < a_n \varepsilon_0 + \varepsilon_1 < 1/8 + 1/8 K \leq 1/4$ and therefore $\|q_{n+1} \beta\| = a_n \varepsilon_0 + \varepsilon_1 \geq \|q_n \beta\|$ for all $n \geq N_0 + 1$ and this is a contradiction since $\|q_{N_0+1} \beta\| \neq 0$. So by Lemma 2

$\beta = \frac{p_{n+1} + \lambda_{n+1} p_n}{q_{n+1} + \lambda_{n+1} q_n}$.

If we define $Q_i$ such, that $|q_i \alpha - Q_i| \leq 1/q_{i+1}$ for all $i$, then (see [4]).

$\lambda_{n+1} = [0; a_{n+1}, a_{n+2}, \ldots] = \frac{Q_{n+1} - \alpha q_{n+1}}{Q_n - \alpha q_n}$

and therefore

$\beta = \frac{(p_{n+1} Q_n - Q_{n+1} p_n) + \alpha (q_{n+1} p_n - q_n p_{n+1})}{(q_{n+1} Q_n - Q_{n+1} q_n)} = a \alpha + b$ with $a, b \in \mathbb{Z}$

because $|q_{n+1} Q_n - Q_{n+1} q_n| = 1$, and the theorem is proved.

PROOF OF THEOREM 2. Let $\gamma$ and $\delta$ irrational be such that

$\delta \neq (a \gamma + b)/(c \gamma + d)$ for all $a, b, c, d \in \mathbb{Z}$

and such that $1, \gamma, \delta$ are linear independent over $\mathbb{Q}$. Therefore not both are of the form $e \alpha + f$ with $e, f \in \mathbb{Z}$. We define now a real number $\alpha$ by its partial quotients $a_0, a_1, \ldots$ such that for the best approximation denominators $q_n$ of $\alpha$ we have $\lim_{n \to \infty} \|q_n \gamma\| = 0$ and $\lim_{n \to \infty} \|q_n \delta\| = 0$.

Let $a_0 = 0$, $q_{-1} = 0$, $q_0 = 1$ and assume that $a_1, a_2, \ldots, a_n$ and therefore $q_1, q_2, \ldots, q_n$ are defined.
$1, q_n \gamma$ and $q_n \delta$ are linearly independent over $\mathbb{Q}$ and therefore the sequence $((kq_n \gamma), (kq_n \delta))_{k \in \mathbb{N}}$ is dense in the unit square. Therefore we can choose an integer $a_{n+1}$ such that

$$\max(||a_{n+1}q_n \gamma + q_{n-1}\gamma||, ||a_{n+1}q_n \delta + q_{n-1}\delta||) \leq 1/n.$$ 

Then with $q_{n+1} = a_{n+1}q_n + q_{n-1}$ we have $\lim_{n \to \infty} ||q_n \beta|| = 0$ for $\beta = \gamma$ and for $\beta = \delta$ and the theorem is proved.
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INSTITUT FÜR MATHEMATIK, HELLBRUNNERSTRASSE 34, A-5020 SALZBURG, AUSTRIA