NEAR-FIELDS ASSOCIATED WITH INVARIANT LINEAR $\kappa$-RELATIONS
PETER FUCHS AND C. J. MAXSON
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ABSTRACT. In this paper we investigate a construction method for subnear-rings of $M(G)$ proposed by H. Wielandt using subgroups of direct powers $G^\kappa$ of $G$ called invariant linear $\kappa$-relations. If $\kappa = 2$ we characterize, in terms of properties of these subgroups, when the associated near-rings are near-fields and prove that every near-field arising from an invariant linear 2-relation must be a field.

I. Introduction. In 1972 H. Wielandt [7] presented a very general method for constructing subnear-rings of the near-ring $M(G)$ of functions on the group $G$. A particular instance of this construction, namely centralizer near-rings, has been extensively investigated in the past several years. In this paper we initiate a study of the structure of the near-rings obtained by Wielandt's general method.

We recall the construction. Let $(G, +)$ be a group, let $\kappa$ be a cardinal number and let $G^\kappa$ denote the direct product of $\kappa$ copies of $G$. We let $M(G)$ act on $G^\kappa$ component-wise. For any subgroup $H$ of $G^\kappa$ we define

$$M(G, \kappa, H) = \{ f \in M(G) | f(H) \subseteq H \}.$$ 

These $M(G, \kappa, H)$ are subnear-rings of $M(G)$ with identity $id: G \to G$, $id(x) = x \forall x \in G$.

One is therefore led to an investigation of the transfer of information between the structure of the near-rings $M(G, \kappa, H)$ and the subgroups $H$ of $G^\kappa$. Wielandt calls these subgroups invariant linear $\kappa$-relations and indicates that these linear $\kappa$-relations might be studied as in his work on permutation groups $P$ via $P$-invariant $\kappa$-relations [6].

Another reason for investigating the near-rings $M(G, \kappa, H)$ is that they are indeed very general as indicated in the following theorem. Let $R$ be a near-ring with identity, 1. It is well known that $R$ can be embedded in $M(G)$ for some group $G$.

**Theorem I.1.** Let $R$ be a near-ring with identity 1. Then there exists a group $G$, a cardinal number $\kappa$, and a subgroup $H$ of $G^\kappa$ such that $R \simeq M(G, \kappa, H)$.

The reader is referred to the books by Meldrum [2] and Pilz [3] for the proof of this result as well as background information on near-rings.

In [4], Remak investigated the subgroup structure of $G^2$ and in [5] indicated how this can be extended to the case $\kappa \geq 3$. We briefly outline his results. Again let $G$ be a group, $\kappa$ a positive integer, $\kappa \geq 2$, and for $j \in \{1, \ldots, \kappa\}$ let $B_j$ be a
subgroup of \( G \), \( \overline{B}_j \) a normal subgroup of \( B_j \) such that \( B_j / \overline{B}_j \simeq B_{j+1} / \overline{B}_{j+1} \) with isomorphisms \( \sigma_j, j \in \{1, \ldots, \kappa-1\} \). Let \( \alpha \) be an ordinal, \( \{b_{1\eta} | \eta < \alpha\} \) a set of coset representatives of \( \overline{B}_1 \) in \( B_1 \) where \( b_{10} = 0 \) and define a subset \( H \subseteq G^\kappa \) by

\[
H = \bigcup_{\eta < \alpha} \left( (b_{1\eta} + \overline{B}_1) \times \prod_{j=1}^{\kappa-1} (\sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_1 (b_{1\eta} + \overline{B}_1)) \right).
\]

\( H \) is called a \( \kappa \)-fold meromorphic product and will be denoted by

\[
H = B_1 / \overline{B}_1 \times \sigma_1 B_2 / \overline{B}_2 \times \cdots \times \sigma_{\kappa-1} B_\kappa / \overline{B}_\kappa.
\]

It is straightforward to verify that \( H \) is a subgroup of \( G^\kappa \) but in general not every subgroup of \( G^\kappa \) is a \( \kappa \)-fold meromorphic product. For \( \kappa = 2 \), however we have such a result.

**Theorem 1.2 (Klein-Fricke) [4].** Every subgroup of \( G \times G \) is a 2-fold meromorphic product.

In this paper we focus on near-fields for the case \( \kappa = 2 \). In the next section we characterize when \( M(G, 2, H) \) is a near-field and find the somewhat surprising result that the only near-fields arising in this case are fields.

**II. When is \( M(G, 2, H) \) a near-field?** We now turn to a characterization of the triples \( (G, 2, H) \) such that \( M(G, 2, H) \) is a near-field. From the Klein-Fricke Theorem we know that \( H = B_1 / \overline{B}_1 \times \sigma_1 B_2 / \overline{B}_2 \). For \( G = \mathbb{Z}_2 \) the subgroups \( H_1 = \mathbb{Z}_2 / \mathbb{Z}_2 \times \{0\} / \{0\} = \mathbb{Z}_2 \times \{0\} \), \( H_2 = \{0\} \times \mathbb{Z}_2 \) and \( H_3 = \{0\} \times \{0\} \) are such that \( M(G, 2, H_i) \simeq \mathbb{Z}_2 \), \( i = 1, 2, 3 \). For \( H_4 = \{(0,0),(1,1)\} \) and \( H_5 = \mathbb{Z}_2 \times \mathbb{Z}_2 \) we get \( M(G, 2, H_4) = M(G, 2, H_5) = M(\mathbb{Z}_2) \) which is not a near-field. For the remainder of the paper we take \( |G| > 2 \) and in a sequence of lemmas show that when \( M(G, 2, H) \) is a near-field, \( H \) has the form \( G \times \sigma G \). For a subgroup \( S \) of \( G \) we let \( S^* \) denote \( S \setminus \{0\} \).

**Lemma II.1.** Let \( H = B_1 / \overline{B}_1 \times \sigma_1 B_2 / \overline{B}_2 \). If \( N = M(G, 2, H) \) is a near-field then \( B_1 = B_2 = G \).

**Proof.** We may assume that \( B_1 \cup B_2 \neq \{0\} \) since otherwise \( M(G, 2, H) = M_0(G) \) is not a near-field. If \( B_1 \cup B_2 \neq G \) then the function \( f : G \to G \) given by \( f(x) = x \) if \( x \in G \setminus (B_1 \cup B_2) \) and \( f(x) = 0 \) if \( x \in B_1 \cup B_2 \) is in \( N \) contradicting the fact that \( N \) is a near-field. Hence \( B_1 \cup B_2 = G \) so at least one of \( B_1, B_2 \) must equal \( G \), say \( B_1 = G \). Suppose \( B_2 \neq G \) and take \( y \in G \setminus B_2 \).

**Case (i).** \( \overline{B}_1 \neq \{0\} \). Let \( \tilde{b}_1 \in \overline{B}_1^* \). One verifies that the function \( h : G \to G \) defined by \( h(y) = \tilde{b}_1 \) and \( h(x) = 0 \) for \( x \neq y \) is in \( N \). Since \( |G| \geq 3 \), \( h \) is not invertible, a contradiction.

**Case (ii).** \( \overline{B}_1 = \{0\} \). Then \( H = G / \{0\} \times \sigma B_2 / \overline{B}_2 \). Now \( \sigma(y) = b_2 + \overline{B}_2 \) for some \( b_2 \in B_2 \setminus \overline{B}_2 \). Define \( A_1 = \{x | x \in b_2 + \overline{B}_2 \} \), \( A_n = \bigcup \{\sigma(x) | x \in A_{n-1}\} \) for \( n \geq 2 \). Let \( A = \bigcup_{n=1}^\infty A_n \). We define \( f : G \to G \) by \( f(x) = 0 \), \( x \in A \cup \{y\} \) and \( f(x) = x \), \( x \notin A \cup \{y\} \) and note that \( f \in N \). If \( 0 \neq y' = y + b_2 \), then \( y' \notin A \cup \{y\} \) since \( A \subseteq B_2 \). Hence \( f(y') = y' \) so \( f \) is not the zero map. Since \( f \) is not invertible, we have a contradiction.

Therefore we must conclude that \( B_1 = B_2 = G \).

Now let \( H = G / B_1 \times \sigma G / B_2 \) and let \( N = M(G, 2, H) \). When \( N \) is a near-field, \( N \) is zero-symmetric since \( N \) contains the identity map and therefore cannot be
isomorphic to the constant maps on \( \mathbb{Z}_2 \). Thus in this case we must have \( B_1 \cap B_2 = \{0\} \), for if \( 0 \neq y \in B_1 \cap B_2 \) then \( f: G \to G, f(x) = y \forall x \in G \) is an element of \( N_c \setminus \{0\} \), a contradiction. Thus \( B_1 \oplus B_2 \) is a normal subgroup of \( G \). We now develop some notation. Let \( \alpha, \beta \) be ordinals such that \( B_1 = \{b_{1\eta} | \eta < \alpha\} \), \( B_2 = \{b_{2\gamma} | \gamma < \beta\} \) where \( b_{10} = b_{20} = 0 \). Further, if \( B_1 \neq \{0\} \), let \( \sigma(y + B_1) = b_{1n} + B_2, 1 \leq n < \alpha \) and \( \sigma(b_{21} + B_1) = x_\gamma + B_2, 1 \leq \gamma < \beta \) if \( B_2 \neq \{0\} \). Define \( X_0 = \emptyset, X_0 = \{x_\alpha + B_2 | 1 \leq \gamma \leq \beta \} \) and for \( n \geq 1 \), \( X_n = \{x+B_1|x+B_1 \cap n+B_2 \neq \emptyset \) for some \( w+B_2 \in X_{n-1}\}, \) and \( \overline{X}_n = \{\sigma(x + B_1)|x + B_1 \in X_n\}. \) Then define \( X = \bigcup_{n=0}^{\infty}(X_n \cup \overline{X}_n) \). If \( B_2 = \{0\} \) define \( X = \emptyset \). In a similar manner let \( Y_0 = \emptyset, Y_0 = \{y_{\eta} + B_1 | 1 \leq \eta < \alpha\} \), and for \( n \geq 1 \), \( Y_n = \{y + B_2 | y + B_2 \cap w + B_1 \neq \emptyset \) for some \( w + B_1 \in Y_{n-1}\}, \) \( \overline{Y}_n = \{\sigma^{-1}(y + B_2)|y + B_2 \in Y_n\}. \) Let \( Y = \bigcup_{n=0}^{\infty}(Y_n \cup \overline{Y}_n) \). If \( B_1 = \{0\} \) define \( Y = \emptyset \). For Lemmas II.2–II.5 we always assume that \( B_1 \neq \{0\} \) and \( B_2 \neq \{0\} \). If either \( B_1 = \{0\} \) or \( B_2 = \{0\} \) then the results are trivial or can be seen to hold by making obvious modifications.

**Lemma II.2.** For \( n \geq 0 \) let \( A_n = \bigcup_{k=0}^{n} \bigcup Y_\kappa \cup B_1 \) and \( G_n = \bigcup_{k=0}^{n} \bigcup \overline{X}_\kappa \cup B_2. \) Then \( A_n \) and \( G_n \) are subgroups of \( G \) for each \( n \geq 0 \).

**Proof.** We show that \( A_n \) is a subgroup of \( G \) for \( n \geq 0 \). A similar argument can be used for \( G_n, n \geq 0 \). Since \( B_1 \oplus B_2 = B_2 \oplus B_1 \) is a subgroup of \( G \), so is \( \bigcup Y_0 \cup B_1 \). Suppose we have shown that \( A_m \) is a subgroup for all \( 0 < m < n \). Let \( z_1, z_2 \in A_n. \)

Case (i). If \( z_1, z_2 \in A_{n-1} \), then \( z_1 - z_2 \in A_{n-1} \subseteq A_n. \)

Case (ii). If \( z_1, z_2 \in \bigcup Y_n \) let \( u_1, u_2 \in \bigcup Y_n \) such that \( u_1, u_2 \in \bigcup Y_{n-1} \) with \( (z_1, w_1) \in H \), \( (z_2, w_2) \in H \). Thus \( (z_1 - z_2, w_1 - w_2) \in H \). If \( w_1 - w_2 = 0 \), then \( z_1 - z_2 \in B_1 \). \( w_1 \in B_1 \), then \( z_1 - z_2 \in \emptyset \). Finally, if \( w_1 - w_2 \in \bigcup Y_i \) for some \( i \leq n - 1 \), then \( w_1 - w_2 \in \bigcup Y_{i+1} \), thus \( z_1 - z_2 \in \bigcup Y_{i+1} \subseteq A_n. \)

Case (iii). If \( z_1 \in B_1 \), \( z_2 \in \bigcup Y_n \), let \( y \in \bigcup Y_n \) such that \( y \in \bigcup Y_{n-1} \) with \( (z_2, y) \in H \). Since \( (z_1, 0) \in H \), \( (z_1 - z_2, -y) \in H \). Now \( -y \in \bigcup Y_i \) for some \( i \leq n - 1 \), hence \( -y \in \bigcup Y_{i+1} \) and \( z_1 - z_2 \in \bigcup Y_{i+1} \subseteq A_n. \)

Case (iv). If \( z_1 \in \bigcup \overline{Y}_n \), \( z_2 \in \bigcup \overline{Y}_n \) let \( w \in \bigcup \overline{Y}_{n-1} \) and \( b_1 \in B_1^* \) such that \( (z_1, w) \in H \), \( (z_2, b_1) \in H \). Thus \( (z_1 - z_2, w - b_1) \in H \). Since \( w - b_1 \in \bigcup \overline{Y}_{n-1} \), \( w_1 \in \bigcup \overline{Y}_{n-1} \), \( z_1 - z_2 \in \bigcup \overline{Y}_n \). \( \bigcup \overline{Y}_n \subseteq A_n. \)

Case (v). Finally let \( z_1 \in \bigcup Y_i \) for some \( n > i > 0 \) and \( z_2 \in \bigcup Y_n \). Let \( w_1 \in \bigcup Y_{i-1} \), \( w_2 \in \bigcup Y_{n-1} \) such that \( (z_1, w_1) \in H \), \( (z_2, w_2) \in H \). Thus \( (z_1 - z_2, w_1 - w_2) \in H \). Since \( w_1 - w_2 \in A_{n-1} \), \( w_1 - w_2 \in B_1 \) or \( w_1 - w_2 \in \bigcup Y_i \) for some \( i \leq n - 1 \). If \( w_1 - w_2 = 0 \) then \( z_1 - z_2 \in B_1 \subseteq A_n \), if \( w_1 - w_2 \in B_1^* \) then \( z_1 - z_2 \in \bigcup \overline{Y}_0 \subseteq A_n \). If \( w_1 - w_2 \in \bigcup Y_i \), then \( w_1 - w_2 \in \bigcup Y_{i+1} \), so \( z_1 - z_2 \in \bigcup \overline{Y}_{i+1} \subseteq A_n. \)

**Lemma II.3.** If \( N = M(G, 2, H) \) is zero-symmetric then \( Y \cap X = \emptyset \).

**Proof.** We first show that \( \bigcup \overline{Y}_0 \cap X = \emptyset \). Suppose \( y \in \bigcup \overline{Y}_0 \cap B_2 \). Let \( b_1 \in B_1^* \) such that \( (y, b_1) \in H \). Since \( y \in B_2 \), \( (b_1, y) \in H \). Thus \( (y + b_1, b_1 + y) = (b_1 + y, b_1 + y) \in H \). Since \( N \) is zero-symmetric \( b_1 + y = 0, \) so \( y = -b_1. \) But \( B_1 \cap B_2 = \{0\} \), hence \( b_1 = 0 \), a contradiction. Consequently \( \bigcup \overline{Y}_0 \cap B_2 = \emptyset \). Suppose that \( y \in \bigcup \overline{Y}_0 \cap \bigcup \overline{X}_0 \). Let \( b_1 \in B_1^*, b_2 \in B_2^* \) such that \( (y, b_1) \in H \), \( (b_2, y) \in H \). Then \( (y + b_2, b_1 + y) \in H \) and \( (y + b_2, b_1 + y) = (y + b_2, y + b_1) \) for some \( b_1 \in B_1^* \). Since \( (b_1, b_2) \in H \), \( (y + b_2, b_1 + y) \in H \) and \( (y + b_2, b_1 + y) = (y + b_1 + b_2, y + b_1 + b_2) \in H. \)
Thus $y + b_1 + b_2 = 0$ and $y + b_1 = -b_2$. Hence $y + b_1 \in \bigcup Y_0 \cap B_2$, a contradiction to our first statement. Therefore $\bigcup Y_0 \cap G_0 = \emptyset$. Suppose that $\bigcup Y_0 \cap G_m = \emptyset$ for all $0 \leq m < n$ and $y \in \bigcup Y_0 \cap G_n$. Then $y \in \bigcup X_n$ and there exists $r_1, \ldots, r_m \in G_{n-1}$ and $b_2 \in B_2^*$ such that $(r_1, y) \in H$, $(r_2, r_1) \in H_\ldots (r_m, r_{m-1}) \in H$ and $(b_2, r_m) \in H$. Therefore $(r_1 + \ldots + r_m + b_2, y + r_1 + \ldots + r_m) \in H$. Since $(y, b_1) \in H$ for some $b_1 \in B_1^*$, $(y + r_1 + \ldots + r_m + b_2, b_1 + y + r_1 + \ldots + r_m + b_2) \in H$. Since $N$ is zero-symmetric we must have that $b_1 + y = -b_2 - r_m - \ldots - r_i$. From the previous Lemma, $-b_2 - r_m - \ldots - r_i \in G_{n-1}$ which implies $b_1 + y \in \bigcup Y_0 \cap G_{n-1}$, a contradiction. Hence $\bigcup Y_0 \cap G_n = \emptyset$, $\forall n \geq 0$. If $y \in \bigcup Y_0 \cup X_n$ for some $n \geq 1$, then for some $b_1 \in B_1$, $y + b_1 \in \bigcup Y_0 \cap G_{n-1}$ contradicting the previous situation. We have now shown that $\bigcup Y_0 \cap X = \emptyset$. Suppose that $\bigcup Y_m \cap X = \emptyset$, $\forall m \leq n$. Let $z \in \bigcup Y_n \cap X$. Then for some $b_1 \in B_1$ and some $n \geq 0$, $z + b_1 \in \bigcup X_n$ and therefore $z + b_1 \in \bigcup Y_n \cap \bigcup X_n$. Let $w \in \bigcup Y_n$ such that $w \in \bigcup Y_{n-1}$ and $(z + b_1, w) \in H$. Since $z + b_1 \in \bigcup X_n$, $z + b_1 \in \bigcup X_{n+1}$. Thus $w \in \bigcup X_{n+1} \cap \bigcup Y_{n-1}$, a contradiction. Consequently $\bigcup Y_n \cap X = \emptyset$, $\forall n \geq 0$. If $z \in \bigcup Y_n \cap X$ for some $n \geq 1$, then $z = z_1 + b_2$ for some $z_1 \in \bigcup Y_{n-1}$, $b_2 \in B_2$ and $z = z_2 + b_1$ for some $z_2 \in \bigcup X_n$, $b_1 \in B_1$. Since $z_1 + b_2 = z_2 + b_1$, $z_1 - b_1 = z_2 - b_2$. But $z_1 - b_1 \in \bigcup Y_{n-1}$ and $z_2 - b_2 \in \bigcup X_n$, a contradiction to $\bigcup Y_{n-1} \cap X = \emptyset$. The result now follows.

**LEMMA II.4.** If $M(G, 2, H)$ is zero-symmetric, then (1) $Y \cap (B_1 \oplus B_2) = \emptyset$.
(2) $X \cap (B_1 \oplus B_2) = \emptyset$.

**PROOF.** (1) We first show that $\bigcup Y_n \cap (B_1 \oplus B_2) = \emptyset$, $\forall n \geq 0$. As in the proof of Lemma II.3 we can see that $\bigcup Y_0 \cap (B_1 \oplus B_2) = \emptyset$, $0 \leq m < n$. If $y = b_1 + b_2$ for some $y \in \bigcup Y_n$, $b_1 \in B_1$, $b_2 \in B_2$ then $b_2 = y - b_1 \in \bigcup Y_n \cap B_2$. If $b_2 \neq 0$ then $Y_n \cap \bigcup X_0 = \emptyset$, a contradiction while if $b_2 = 0$ then $y = b_1 \in B_1$ which implies that $B_2 \in Y_n$ and therefore $B_2 \cap \bigcup Y_{n-1} = \emptyset$, a contradiction to our assumption. Thus $\bigcup Y_n \cap (B_1 \oplus B_2) = \emptyset$, $\forall n \geq 0$. If $b_1 + b_2 \in \bigcup Y_n$ for some $n \geq 1$, then for some $b_2 \in B_2$, $b_1 + b_2 \in \bigcup Y_{n-1}$, a contradiction. This establishes (1). The second statement can be shown in a similar way.

**LEMMA II.5.** If $M(G, 2, H)$ is zero-symmetric, then
(1) $\bigcup Y_n \cap \bigcup Y_{k=0}^n \cup Y_n = \emptyset$, $\forall n \geq 0$,
(2) $\bigcup X_n \cap \bigcup X_{k=0}^n \cup X_n = \emptyset$, $\forall n \geq 0$.

**PROOF.** (1) Let $n$ be minimal so that $\bigcup Y_n \cap \bigcup Y_{k=0}^n \cup Y_n \neq \emptyset$. Obviously $n \geq 1$. Then $\bigcup Y_m \cap \bigcup Y_{k=0}^m Y_n = \emptyset$ for all $0 \leq m < n$. Let $y \in \bigcup Y_n \cap \bigcup Y_{k=0}^n \cup Y_k$, say $y \in Y_j$ for some $1 \leq j \leq n$. Suppose that $y \neq \bigcup Y_{j-1}$. Let $x \in \bigcup Y_n$ such that $x \in \bigcup Y_{n-1}$ and $(y, x) \in H$. If $j - 1 \neq 0$ then $x \in \bigcup Y_{n-1} \cap \bigcup Y_{j-1}$, a contradiction. If $j - 1 = 0$, then $x \in B_1 \oplus B_2 \cap \bigcup Y_{n-1}$ which contradicts Lemma II.4. Consequently $y \notin \bigcup Y_{j-1}$, so $y + b_2 \in \bigcup Y_{j-1}$ for some $b_2 \in B_2$. But then $-(y + b_2) + y = -b_2 - y + y = -b_2 \in A_n$. By Lemma II.4 $-b_2 \in B_1^*$, a contradiction since $B_1 \oplus B_2 = \{0\}$. The other statement follows similarly.

The finite case now follows from the above lemmas.
THEOREM II.6. Let $G$ be a finite group, $|G| \geq 3$ and $H = G/B_1 \times \sigma G/B_2$. If $M(G,2,H)$ is zero-symmetric, then $B_1 = \{0\} = B_2$.

PROOF. Suppose that $B_1 \neq \{0\}$. Then $B_2 \neq \{0\}$ and since $G$ is finite there exists a positive integer $n$ such that $\bigcup \nabla_n \cap \bigcup \kappa = \emptyset$. The result now follows from Lemma II.5.

From Theorem II.6 we note that when $G$ is a finite group and $H = G/B_1 \times \sigma G/B_2$ with $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$ then there exists $x \in G^*$ such that $(x,x) \in H$. This implies that every group $H$ of this form contains a nontrivial subgroup of the diagonal $\{(x,x)|x \in G\}$. Therefore the associated near-ring $M(G,2,H)$ cannot be zero-symmetric. We now present an example to show that this is not the case when $G$ is infinite.

EXAMPLE II.7. Let $\{0\} \neq A$ be a group with identity $0$, $G = \bigoplus_{z \in \mathbb{Z}} A = \{(x_z)_{z \in \mathbb{Z}} \in A^\mathbb{Z}|x_z = 0$ for all but finitely many $z \in \mathbb{Z}\}$. Further let $B_1 = \{(x_z)_{z \in \mathbb{Z}} \in G|x_z = 0, \forall z \neq 0\}$ and $B_2 = \{(x_z)_{z \in \mathbb{Z}} \in G|x_z = 0, \forall z \neq 1\}$. Define $\phi: G/B_1 \rightarrow G/B_2$ by $\phi((x_z)_{z \in \mathbb{Z}} + B_1) = (y_z)_{z \in \mathbb{Z}} + B_2$ where $y_z = x_{z-1} \forall z \in \mathbb{Z}$.

It is straightforward to verify that $\phi$ is an isomorphism, so $\phi$ determines a 2-fold meromorphic product $H = G/B_1 \times_{\phi} G/B_2$ with $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$. One can check that $(x_z)_{z \in \mathbb{Z}} + B_1 \cap \phi((x_z)_{z \in \mathbb{Z}} + B_1) = \emptyset$ for all $(x_z)_{z \in \mathbb{Z}} \in G \setminus B_1$. Since $B_1 \cap B_2 = \{0\}$ it follows that there is no $x \in G^*$ such that $(x,x) \in H$. Thus $M(G,2,H)$ is zero-symmetric.

We are now ready to establish our major result.

THEOREM II.8. Let $G$ be an arbitrary group, $|G| \geq 3$ and $H = G/B_1 \times \sigma G/B_2$. If $N = M(G,2,H)$ is a near-field, then $B_1 = \{0\} = B_2$.

PROOF. Case (A): We first suppose that $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$. We may also assume that there is no $0 \not= x \in G$ with $(x,x) \in H$. We define a function $f: G \rightarrow G$ as follows.

(i) Let $f(0) = 0$, $f(b_{1\eta}) = b_{11}$ for all $1 \leq \eta < \alpha$, $f(b_{2\eta}) = b_{21}$ for all $1 \leq \gamma < \beta$ and if $b_{1\eta} + b_{2\gamma} \in B_1 \oplus B_2$, $1 \leq \eta < \alpha$, $1 \leq \gamma < \beta$ let $f(b_{1\eta} + b_{2\gamma}) = b_{11} + b_{21}$.

(ii) If $b_2 \in B_2$ define $f(x_1 + b_2) = x_1$. For $2 \leq \gamma < \beta$ let $f(x_\gamma) = x_1$ and $f((x_1 + b_2) = x_1 + b_{11}$ if $b_2 \in B_2^*$. Thus $f$ is defined on $\bigcup \nabla_m$.

Suppose we have defined $f$ on $\bigcup X_m$ and $\bigcup \overline{X}_m$ for each $0 < m < n$.

(iii) For $x \in \bigcup \overline{X}_{n-1}$ and $b_1 \in B_1^*$ let $f(x + b_1) = b_{11} + f(x)$. This defines $f$ on $\bigcup X_n$.

(iv) In order to define $f$ on $\bigcup \overline{X}_n$ we choose an arbitrary but fixed set of coset representatives $\{w_\xi|\xi < \delta_n\}$ of $\overline{X}_n$ such that for each $z_\xi + B_1 \in X_n$, $\sigma(z_\xi + B_1) = w_\xi + B_2$. If $\sigma(f(z_\xi) + B_1) = w_\xi + B_2$ we define $f(w_\xi + b_2) = w_\xi + b_{11}$ if $b_2 \in B_2^*$ and $f(w_\xi) = w_\xi$.

In a similar manner we now define $f$ on $Y$.

(v) If $b_1 \in B_1$ let $f(y_1 + b_1) = y_1$. If $2 \leq \xi < \alpha$ let $f(y_\eta) = y_1$ and $f(y_\eta + b_1) = y_1 + b_{11}$ for $b_1 \in B_1^*$.

Suppose $f$ has been defined on $\bigcup Y_m$ and $\bigcup \overline{Y}_m$ for all $0 \leq m < n$.

(vi) For all $y \in \bigcup \overline{Y}_{n-1}$ and $b_2 \in B_2^*$ let $f(y + b_2) = f(y) + b_{21}$.

(vii) Choose an arbitrary but fixed set of coset representatives $\{w_\xi|\xi < \delta_n\}$ of $\overline{Y}_n$ such that for $z_\xi + B_2 \in \bigcup Y_n$, $\sigma^{-1}(z_\xi + B_2) = w_\xi + B_1$. If $\sigma^{-1}(f(z_\xi) + B_2) = w_\xi + B_1$ define $f(w_\xi) = w_\xi$ and $f(w_\xi + b_1) = w_\xi + b_{11}$ for $b_1 \in B_1^*$.
We have now defined \( f \) on \( X \cup Y \cup B_1 \oplus B_2 \).

(viii) Let \( S = \{ y + x | y \in \bigcup Y_n \text{ for some } n \geq 0, x \in \bigcup X_m \text{ for some } m \geq 0 \} \). For \( y + x \in S \) let \( f(y + x) = f(y) + f(x) \).

(ix) Finally define \( f(z) = z \) if \( z \notin X \cup Y \cup S \cup B_1 \oplus B_2 \).

We now show that \( f \in N \).

1: \( f \) is well defined.

(i) Since \( B_1 \cap B_2 = \{0\} \), \( f \) is well defined on \( B_1 \oplus B_2 \).

(ii) Let \( n \geq 1 \). We need to show that \( f \) is well defined on \( X \).

Let \( \{ w_\xi | \xi < \delta \} \) be a set of representatives for the cosets in \( \bigcup X_{\kappa-1} \) and let \( y \in \bigcup X_n \). Then \( y \) has the form \( y = x + b_1 \) for some \( x \in \bigcup X_{\kappa-1}, b_1 \in B_1 \). Suppose \( y = w_{\xi_1} + b_1 + b_2 = w_{\xi_2} + b_1' + b_1'' \), where \( b_1, b_1' \in B_1, b_2, b_1'' \in B_2, \xi \neq \xi' \). Then \( -b_2 - w_{\xi_1} + w_{\xi_2} + b_1' - b_1 \in G_{\kappa-1} \cap B_1 \). By Lemma II.4 and since \( B_1 \cap B_2 = \{0\} \) this can only happen if \( -b_2 - w_{\xi_1} + w_{\xi_2} + b_1' - b_1 = 0 \). But then \( w_{\xi_1} + b_2 = w_{\xi_2} + b_1' \), a contradiction. By Lemma II.5, \( f \) is well defined on \( X \).

(iii) Similar arguments show that \( f \) is well defined on \( Y \).

(iv) We show that \( f \) is well defined on \( S \). Suppose that \( y_1 + x_1 = y_2 + x_2 \), \( y_1 \in \bigcup Y_j, y_2 \in \bigcup Y_i, x_1 \in \bigcup X_m, x_2 \in \bigcup X_n \). Then \( -y_2 + y_1 = x_2 - x_1 \in Y \cup B_1 \cap X \cup B_2 \) by Lemma II.2. According to Lemmas II.3 and II.4 this implies \( y_1 = y_2 \) and \( x_1 = x_2 \).

It is easy to show from Lemmas II.3 and II.4 that \( S \cap X = \emptyset, S \cap Y = \emptyset \) and that \( S \cap B_1 \oplus B_2 = \emptyset \). Suppose, for example, \( y + x \in \bigcup X_k \cap S \) for some \( \kappa \geq 1 \). Then \( y + x = x' + b_1 \) for some \( x' \in \bigcup X_{\kappa-1}, b_1 \in B_1 \). Then \( y + x - b_1 = x' \), so \( y + b_1 = x' - x \in Y \cap (X \cup B_2) \) for some \( b_1 \in B_1 \). This contradicts Lemmas II.3 and II.4.

Lemmas II.3 and II.4 now show that \( f \) is well defined on \( G \).

2: \( f \in N \). Let \( (x, y) \in H \). We must show that \( f((x, y)) = (f(x), f(y)) \in H \).

Case (i) If \( x \in B_1 \), then \( y \in B_2 \), so \( f((x, y)) \in H \).

Case (ii) If \( x = b_2 + b_1 \), \( 1 \leq \gamma < \beta \), \( b_1 \in B_1 \), then \( y = x_\gamma + b_2 \) for some \( b_2 \in B_2 \).

Now \( f(x) = b_2 \) or \( f(x) = b_2 + b_1 + b_21 \), and \( f(y) = x_1 \) or \( f(y) = x_1 + b_21 \). In any case \( (f(x), f(y)) \in H \).

Case (iii) Let \( x \in X \). If \( x \in \bigcup X_k \) for some \( \kappa \geq 1 \), then \( y \in \bigcup X_k \). We have that \( x = x' + B_1 \) for some \( x' \in \bigcup X_{\kappa-1} \). By construction of \( f \), \( f(x) \in f(x') + B_1 \) and \( f(y) \in \sigma f(x') + B_1 \). Thus \( (f(x), f(y)) \in H \). If \( x \in \bigcup X_k \) for some \( \kappa \geq 0 \), then \( x \in \bigcup X_{\kappa+1} \) and we are back to the previous case.

Case (iv) Let \( x \in Y \). If \( x = y_\eta + b_1 \in \bigcup Y_0 \), then \( y = b_1 + y_\eta \), \( 1 \leq \eta < \alpha \) for some \( b_1 \in B_2 \). Now \( f(x) = y_1 \) or \( f(x) = y_1 + b_21 \) and \( f(y) = b_1 + y_1 \) or \( f(y) = b_1 + b_21 \). In any case \( (f(x), f(y)) \in H \). Let \( x \in \bigcup Y_n \) for some \( n \geq 1 \). Then \( y \in \bigcup Y_n \) and with arguments similar to those in case (iii), we can show that \( (f(x), f(y)) \in H \). Let \( x \in \bigcup Y_1 \). If \( x \in \bigcup Y_0 \) we have just seen that \( (f(x), f(y)) \in H \). Hence we may assume that \( x = x' + b_2 \gamma \) for some \( x' \in \bigcup Y_0, 1 \leq \gamma < \beta \). Let \( 1 \leq \eta < \alpha \) so that \( (x', b_1) \in H \). Hence \( y = b_1 + x_\eta + b_2 \) for some \( b_2 \in B_2 \). Now \( f(x) = f(x' + b_2 \gamma) = f(x') + b_2 \) and \( f(x') + b_2 = f(x') + b_21 \in y_1 + b_21 + b_2 \). Further \( f(y) = f(y_1 + x_\gamma + b_2) = b_21 + f(x_\gamma + b_2) \) and \( b_21 + f(x_\gamma + b_2) = b_21 + x_1 + b_21 \) or \( b_21 + f(x_\gamma + b_2) = b_21 + x_1 + b_21 \). In any case \( (f(x), f(y)) \in H \).

Finally if \( x \in \bigcup Y_n \) for some \( n \geq 2 \), then \( x = x' + b_2 \gamma \) for some \( x' \in \bigcup Y_{n-1} \), \( b_2 \in B_2 \). We may assume that \( b_2 = b_2 \gamma \in B_2 \). Let \( y' \in \bigcup Y_{n-1} \) such that \( y' \in \bigcup Y_{n-2} \) with \( (x', y') \in H \). Then \( (x' + b_2 \gamma, y' + x_\gamma) \in H \), thus \( y = y' + x_\gamma + b_2 \).
for some $b_2 \in B_2$. Since $y \in S$, $(f(x), f(y)) = (f(x') + b_2, f(y') + f(x_2 + b_2))$ is
either equal to $(f(x') + b_2, f(y') + x_1 + b_2)$ or equal to $(f(x') + b_1, f(y') + x_1)$. Since
$(f(x'), f(y')) \in H$ as shown previously and $(b_2, x_1) \in H$ we have $(f(x), f(y)) \in H$.

Case (v) Let $x \in S$, say $x = y^* + x^*$, $y^* \in \bigcup \Sigma_{n-1}$, $x^* \in \bigcup \Sigma_{m+1}$
Suppose that $n \geq 1$. Let $y \in \bigcup \Sigma_n$ such that $y \in \bigcup \Sigma_{n-1}$ with $(y^*, y) \in H$ and $x \in \bigcup \Sigma_{m+1}$ such
that $(x^*, x) \in H$. Then $(y^* + x^*, y + x) \in H$. Thus $y = y^* + x^* + b_2$ for
some $b_2 \in B_2$ and $(f(x), f(y)) = (f(y^*) + f(x^*), f(y) + f(x + b_2)) \in H$ according
to Cases (iii) and (iv). If $n = 0$, $y^* \in \bigcup \Sigma_0$. Let $b_1n \in B_1$ with $(y^*, b_1n) \in H$. Then
$(y^* + x^*, b_1n + x) \in H$. Thus $y = b_1n + x + b_2$ for some $b_2 \in B_2$. Hence
$(f(x), f(y)) = (f(y^*) + f(x^*), b_1n + f(x + b_2)) \in H$, since $(f(x^*), f(x + b_2)) \in H$.

Case (vi) Let $x \in \mathbb{G} \setminus (\bigcup \Sigma \cup \Sigma \Sigma B_1 \cup B_2)$. Then clearly $y \notin B_1 \cup B_2$, $y \notin \bigcup \Sigma_{\kappa}$
for any $\kappa \geq 0$, $y \notin Y$. Suppose $y \in \bigcup \Sigma_{n-1}$ for some $\kappa \geq 1$. If $\kappa = 1$, then $y = b_1 + y$
for some $y \in \bigcup \Sigma_0$. Since $y \notin \bigcup \Sigma_0$, $b_1 \neq b_1n$. Now $(b_2, y) \in \Sigma$ for some
$b_2 \in B_2$. Hence $(y + b_2, b_1 + y) \in H$. Thus $x = y + b_2 + b_1$ for some $b_1 \in B_1$. Hence
$x \in \bigcup \Sigma_1$, a contradiction. As $\kappa \geq 2$ let $y = b_1n + y$ for some $y \in \bigcup \Sigma_{\kappa-1}$,
$1 \leq \eta < \alpha$. Let $x \in \bigcup \Sigma_{\kappa-2}$ such that $(x, y) \in H$. Then $x = y + b_2 + b_1$ for $\kappa \geq 1$.
Consequently $(f(x), f(y)) = (x, y) \in H$.

Now follows that $f \in N$. Clearly $f$ is not invertible, since $B_2 \neq \{0\}$ by
assumption and $f(x_2 + b_2) = x$, $\forall b_2 \in B_2$.

Case (B). Suppose that either $B_1 = \{0\}$ or $B_2 = \{0\}$. W.l.o.g. we may assume that
$B_2 = \{0\}$, $B_1 \neq \{0\}$. Then $X = \varnothing$, $S = \varnothing$, $B_1 \cup B_2 = B_1$. Define the function
$f: \mathbb{G} \rightarrow \mathbb{G}$ on $Y \cup B_1$ in the same way as before, noting that $\bigcup \Sigma_n = \bigcup \Sigma_{n-1}$
for all $n \geq 1$. For $z \notin Y \cup B_1$ let $f(z) = z$. As in Case (A) one can verify that $f$ is well
defined, $f \in N$ but $f$ is not invertible since $f(y_1 + b_2) = y_1$ for all $b_1 \in B_1$.

In both cases we obtain a noninvertible function $f \in M(\mathbb{G}, 2, H)$, a contradiction
to $N$ being a near-field. Thus $B_1 = \{0\} = B_2$.

We now show that the only near-fields arising in the case $\kappa = 2$ are fields. From
the discussion prior to Lemma II.1 we see this is the case for $|\mathbb{G}| = 2$. We now turn
to $|\mathbb{G}| > 2$. If $A$ is a group of automorphisms of $\mathbb{G}$, then $A^\circ$ denotes the group with
the zero map adjoined.

**Theorem II.9.** Let $|\mathbb{G}| \geq 3$. If $N = M(\mathbb{G}, 2, H)$ is a near-field then $N$ is a
field. In fact, $N = M_{A^\circ}(G)$ where $A = \langle \alpha \rangle$ is a cyclic group of automorphisms of
$G$ such that $G = X \times \{0\}$ for $x \in G^*$. Conversely, if $B = \langle \beta \rangle$ is a cyclic group
of automorphisms of $G$ such that $G = X \times \{0\}$ for $x \in G^*$, then $M_{B^\circ}(G)$ is a
near-field and $M_{B^\circ}(G) = M(\mathbb{G}, 2, H)$ where $H = G \times \beta G$.

**Proof.** From Lemma II.1 and Theorem II.8, $H = G \times \alpha G = \{(x, \alpha(x))|x \in G\}$. But then $f \in N$ if and only if $\alpha f(x) = f(\alpha x)$ for all $x \in G$, i.e., if and only if $f$ belongs to the centralizer near-ring $M(\alpha)^c(G)$. Since $N$ is a near-field it is well
known (see [1]) that $G^*$ must be the only nonzero orbit for $A = \langle \alpha \rangle$. Finally since $A$ is abelian, $M_{A^0}(G)$ is a field. For the converse it is clear that $M_{B^0}(G) = M(G, 2, H)$ with $H = G \times \beta G$ and again from [1] we see that $M(G, 2, H)$ is a near-field since $((\beta), G)$ satisfies the needed finiteness condition.

**COROLLARY II.10.** If $F$ is a finite field then $F \simeq M(G, 2, H)$ for some group $G$ and some 2-fold meromorphic product $H$.

**PROOF.** We know $F \simeq M_F(F)$ and $F^* = \langle \alpha \rangle$. By the above theorem, $F \simeq M(F, 2, F \times \alpha F)$.

**COROLLARY II.11.** Let $|G| \geq 3$. $H$ is an invariant linear 2-relation of the form $G \times \sigma G$ with $\langle \sigma \rangle$ transitive on $G^*$ if and only if $M(G, 2, H)$ is a field.

We conclude the paper with an example to show that the situation is quite different for $\kappa = 3$. Indeed, we see that one can have a meromorphic product $H = G/B_1 \times \sigma_1 G/B_2 \times \sigma_2 G/B_3$ such that $N = M(G, 3, H)$ is a near-field but $B_1 \neq \{0\}$, $B_2 \neq \{0\}$ and $B_3 \neq \{0\}$.

**EXAMPLE II.12.** Let $G = \mathbb{Z}_2^3$ with the usual basis $\{e_1, e_2, e_3, e_4\}$ and let $B_1 = B_2 = B_3 = G$. Let $\overline{B}_1 = \langle e_1 + e_2, e_3 + e_4 \rangle$, $\overline{B}_2 = \langle e_1, e_1 + e_3 + e_4 \rangle$ and $\overline{B}_3 = \langle e_1, e_1 + e_2 + e_3 \rangle$. The following scheme determines a meromorphic product $H:

\begin{align*}
\overline{B}_1 &\mapsto \overline{B}_2 \mapsto \overline{B}_3, \\
e_1 + \overline{B}_1 &\mapsto e_4 + \overline{B}_2 \mapsto e_1 + e_2 + e_4 + \overline{B}_3, \\
e_1 + e_4 + \overline{B}_1 &\mapsto e_1 + e_2 + \overline{B}_2 \mapsto e_1 + e_4 + \overline{B}_3, \\
e_4 + \overline{B}_1 &\mapsto e_2 + e_3 + \overline{B}_2 \mapsto e_2 + \overline{B}_3.
\end{align*}

One can check that $M(G, 3, H) = \{0, \text{id}\}$.

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**DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843 (Current address of C. J. Maxson)**

*Current address* (Peter Fuchs): Department of Mathematics, Johannes Kepler Universität, A-4040 Linz, Austria