ASYMPTOTIC BEHAVIOR OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS OF nTH ORDER

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ABSTRACT. In this paper we obtain a result of the asymptotic behavior of the nth order equation \( u^{(n)} + f(t,u,u',\ldots,u^{(n-1)}) = 0 \) under some assumptions. For \( n = 2 \) and \( f(t,u,u') \equiv f(t,u) \), it revises the result given by Jingcheng Tong, which is not true in general.

Much work has been done on the asymptotic behavior of the second order equation

\[ u'' + f(t,u) = 0. \]

Some results of it are based on the integral inequalities of Gronwall-Bihari type. Here we quote the Theorem B in [1] as a proposition, according to the author, which includes the Theorem in [2] as its special case.

PROPOSITION. Let \( f(t,u) \) be continuous on \( D: t \geq 0, -\infty < u < \infty \). If there are two nonnegative and continuous functions \( v(t), \phi(t) \) for \( t \geq 0 \), and a continuous function \( g(u) \) for \( u > 0 \), such that

1. \( \int_1^\infty v(t)\phi(t) dt < \infty \),
2. for \( u > 0 \), \( g(u) \) is positive and nondecreasing,
3. \( |f(t,u)| < v(t)\phi(t)g(|u|/t) \) for \( t \geq 1, -\infty < u < \infty \),

then the equation (1) has solutions which are asymptotic to \( a + bt \), where \( a, b \) are constants and \( b \neq 0 \).

According to the proof it seems to be true that every solution \( u(t) \) of Equation (1) satisfies that \( u'(t) \to b \) as \( t \to \infty \). But unfortunately, the Proposition does not hold in general because of two mistakes in its proof.

The first error arises because \( G(c_3) + \int_1^t v(s)\phi(s) ds \) may be outside the domain of \( G^{-1} \) (see (4) in [1]). Subsequently, the second error arises because one cannot hence conclude that \( \int_1^\infty |f(s,u(s))| ds < \infty \). For example, the equation \( u'' - (2/t^4)u^2 = 0 \) has a solution \( u = t^2 \) which does not satisfy the property \( u'(t) \to b \) \((t \to \infty)\) as depicted in the proof of the Proposition. For the same reason, the example in [1] cannot be true.

In this paper we obtain a result of the asymptotic behavior of the nth order equation

\[ u^{(n)} + f(t,u,u',\ldots,u^{(n-1)}) = 0. \]

In a special case, for \( n = 2 \) and \( f(t,u,u') \equiv f(t,u) \), it revises the result in [1].
Here we cite a definition of a function class $\mathcal{F}$ given in [3, p. 207].

**DEFINITION.** A function $g(u)$ is said to belong to $\mathcal{F}$ if $g(u) > 0$ is nondecreasing and continuous on $(-\infty, \infty)$, and

$$g(u)/v \leq g(u/v) \quad \text{for } u \geq 0, \ v \geq 1. \tag{3}$$

It is easy to see that $g(u) \in \mathcal{F}$ implies $\int_1^\infty (1/g(s)) \, ds = \infty$. In fact, from (3), letting $v = u \geq 1$, we get $g(u)/u \leq g(1)$, i.e., $g(u) \leq g(1)u$, which follows

$$\int_1^\infty \frac{ds}{g(s)} \geq \int_1^\infty \frac{ds}{g(1)s} = \infty. \tag{4}$$

At the following we give an extension of the basic Bihari's inequality as a preliminary knowledge. It slightly modifies the Lemma in [4], which is not true for the case that $f(x) \geq 1$ is not satisfied.

**LEMMA.** Assume that $f(t) > 0$ is nondecreasing and continuous on $[t_0, \infty)$, $h(t) \geq 1$ and $\phi(t) \geq 0$ are continuous on $[t_0, \infty)$, and $g(u) \in \mathcal{F}$. Let

$$u(t) \leq f(t) + h(t) \int_{t_0}^t \phi(s)g(u(s)) \, ds \tag{5}$$

for $t \geq t_0$. Then

$$u(t) \leq \frac{f(t)h(t)}{f(t_0)} G^{-1} \left( G(f(t_0)) + \int_{t_0}^t \phi(s)h(s) \, ds \right) \tag{6}$$

for $t \geq t_0$, where

$$G(u) = \int_{u_0}^u \frac{ds}{g(s)} \quad \text{for } u_0 > 0, \ u > 0. \quad \text{(18)}$$

**PROOF.** Without loss of generality, we assume that $u(t) \geq 0$. For otherwise, let $u_1 = \max\{u(x), 0\}$. Then $u_1(t)$ satisfies (5), and $u(t) \leq u_1(t)$. In view of that $f(t)$ is nondecreasing and $h(t) \geq 1$, from (5) we see

$$\frac{u(t)}{f(t)h(t)} \leq 1 + \int_{t_0}^t \phi(s) \frac{1}{f(s)} g(u(s)) \, ds,$$

which follows

$$\frac{f(t_0)}{f(t)h(t)} u(t) \leq f(t_0) + \int_{t_0}^t \phi(s)h(s) \frac{f(t_0)}{f(s)h(s)} g(u(s)) \, ds$$

$$\leq f(t_0) + \int_{t_0}^t \phi(s)h(s) \left( \frac{f(t_0)}{f(s)h(s)} u(s) \right) \, ds.$$

By using Bihari's inequality we get

$$\frac{f(t_0)}{f(t)h(t)} u(t) \leq G^{-1} \left( G(f(t_0)) + \int_{t_0}^t \phi(s)h(s) \, ds \right).$$

Then (6) is true. Noting $\int_1^\infty (1/g(s)) \, ds = \infty$, we know that

$$G(f(t_0)) + \int_{t_0}^t \phi(s)h(s) \, ds \in \text{Dom}(G^{-1}) \quad \text{for } t \geq t_0.$$
THEOREM. Assume \( f(t, u_0, u_1, \ldots, u_{n-1}) \) is continuous on \([1, \infty) \times (-\infty, \infty)^n\), and

\[
|f(t, u_0, u_1, \ldots, u_{n-1})| \leq \sum_{i=0}^{n-1} \phi_i(t)g_i\left(|u_i|/t^{n-i-1}\right)
\]

for \( t \geq 1 \) and \(-\infty < u < \infty\), where \( \phi_i(t) \geq 0 \) (\( i = 0, 1, \ldots, n-1 \)) is continuous on \([1, \infty)\), and \( \int_1^{\infty} \phi_i(t) \, dt < \infty \); \( g_i(u) \in \mathcal{F} \). Then every solution \( u(t) \) of equation (2) satisfies \( u^{(n-i)}(t)/t^{i-1} \to a_i \in \mathbb{R} \) (\( i = 1, 2, \ldots, n \)) as \( t \to \infty \), where \( u^{(0)}(t) = u(t) \). Furthermore, if \( f(t, u_0, u_1, \ldots, u_{n-1}) \) does not change its sign when \( u_i > 0 \) (\( i = 1, 2, \ldots, n-1 \)) and \( t \geq 1 \), then equation (2) has solutions such that \( a_i > 0 \) (\( i = 1, 2, \ldots, n \)).

PROOF. We denote that \( G_i(u) = \int_1^u ds/g_i(s) \).

(i) Noting that \( \int_1^{\infty} \phi_i(s) \, ds < \infty \), according to the Lemma, we can prove by induction that every solution \( u(t) \) of equation (2) satisfies

\[
\left|u^{(n-r)}(t)\right| \leq c_{n-r} + d_{n-r} \int_1^t \sum_{i=0}^{n-r-1} \phi_i(s)g_i\left(|u^{(i)}(s)|/s^{n-i-1}\right) \, ds, \quad r = 1, 2, \ldots, n-1
\]

for \( t \geq 1 \), where \( c_{n-r} > 0 \), \( d_{n-r} > 0 \).

In fact, from (2) we have

\[
u^{(n-1)}(t) = c - \int_1^t f(s, u(s), u'(s), \ldots, u^{(n-1)}(s)) \, ds.
\]

Choose \( c' \) such that \( c' > |c| \) and \( c' > 0 \). From (7) we get

\[
|u^{(n-1)}(t)| \leq \left[ c' + \int_1^t \sum_{i=0}^{n-2} \phi_i(s)g_i\left(|u^{(i)}(s)|/s^{n-i-1}\right) \, ds \right]
\]

\[
+ \int_1^t \phi_{n-1}(s)g_{n-1}\left(|u^{(n-1)}(s)|\right) \, ds.
\]

Using the Lemma we get

\[
|u^{(n-1)}(t)| \leq \left[ 1 + \frac{1}{c'} \int_1^t \sum_{i=0}^{n-2} \phi_i(s)g_i\left(|u^{(i)}(s)|/s^{n-i-1}\right) \, ds \right]
\]

\[
x G_{n-1}^{-1}\left(G_{n-1}(c') + \int_1^t \phi_{n-1}(s) \, ds\right)
\]

\[
\leq c_{n-1} + d_{n-1} \int_1^t \sum_{i=0}^{n-2} \phi_i(s)g_i\left(|u^{(i)}(s)|/s^{n-i-1}\right) \, ds,
\]

where

\[
c_{n-1} = G_{n-1}^{-1}\left(G_{n-1}(c') + \int_1^\infty \phi_{n-1}(s) \, ds\right) > 0, \quad d_{n-1} = c_{n-1}/c' > 0.
\]

Thus (8) is true for \( r = 1 \). Suppose (8) is true for \( r = k < n - 1 \), i.e.,

\[
|u^{(n-k)}(t)| \leq c_{n-k} + d_{n-k} \int_1^t \sum_{i=0}^{n-k-1} \phi_i(s)g_i\left(|u^{(i)}(s)|/s^{n-i-1}\right) \, ds,
\]

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where $c_{n-k} > 0$, $d_{n-k} > 0$. Then

$$|u^{(n-k)}(t)| \leq c_{n-k} t^{k-1} + d_{n-k} t^{k-1} \int_1^t \sum_{i=0}^{n-k-1} \phi_i(s) g_i \left( \frac{|u(i)(s)|}{s^{n-i-1}} \right) \, ds.$$  

Considering that

$$|u^{(n-k)}(t)| - |u^{(n-k)}(1)| \leq \int_1^t |u^{(n-k)}(s)| \, ds,$$  

we have

$$|u^{(n-k-1)}(t)| \leq c'_{n-k} t^k + c''_{n-k} + d''_{n-k} \int_1^t (t^k - s^k) \sum_{i=0}^{n-k-1} \phi_i(s) g_i \left( \frac{|u(i)(s)|}{s^{n-i-1}} \right) \, ds,$$  

where

$$c'_{n-k} = \frac{c_{n-k}}{k}, \quad c''_{n-k} = |u^{(n-k-1)}(1)| - \frac{c_{n-k}}{k}, \quad d''_{n-k} = \frac{d_{n-k}}{k}.$$  

Let $c'''_{n-k} = c'_{n-k} + |c''_{n-k}|$, then

$$\frac{|u^{(n-k-1)}(t)|}{t^k} \leq \left[ c'''_{n-k} + d''_{n-k} \int_1^t \sum_{i=0}^{n-k-2} \phi_i(s) g_i \left( \frac{|u(i)(s)|}{s^{n-i-1}} \right) \, ds \right]$$  

$$+ d''_{n-k} \int_1^t \phi_{n-k-1}(s) g_{n-k-1} \left( \frac{|u^{(n-k-1)}(s)|}{s^{k}} \right) \, ds$$  

for $t \geq 1$. Using the Lemma we get

$$\frac{|u^{(n-k-1)}(t)|}{t^k} \leq \left[ 1 + d''_{n-k} \int_1^t \sum_{i=0}^{n-k-2} \phi_i(s) g_i \left( \frac{|u(i)(s)|}{s^{n-i-1}} \right) \, ds \right]$$  

$$\times G_{n-k-1}^{-1} \left( G_{n-k-1}(c'''_{n-k}) + d''_{n-k} \int_1^t \phi_{n-k-1}(s) \, ds \right)$$  

$$\leq c_{n-k-1} + d_{n-k-1} \int_1^t \sum_{i=0}^{n-k-2} \phi_i(s) g_i \left( \frac{|u(i)(s)|}{s^{n-i-1}} \right) \, ds,$$  

where $c_{n-k-1} = G_{n-k-1}^{-1} (G_{n-k-1}(c'''_{n-k}) + d''_{n-k} \int_1^\infty \phi_{n-k-1}(s) \, ds) > 0$, $d_{n-k-1} = d''_{n-k} c_{n-k-1}/c'''_{n-k} > 0$. Thus (8) is true for $r = k + 1$.

(ii) By induction we can also prove that

$$|u^{(i)}(t)|/t^{n-i-1} \leq b_i, \quad i = 0, 1, \ldots, n - 1,$$  

for $t \geq 1$, where $b_i$ ($i = 0, 1, \ldots, n - 1$) are constants. In fact, from (8), letting $r = n - 1$, we get

$$\frac{|u'(t)|}{t^{n-2}} \leq c_1 + d_1 \int_1^t \phi_0(s) g_0 \left( \frac{|u(s)|}{s^{n-1}} \right) \, ds,$$  

which follows

$$|u'(t)| \leq c_1 t^{n-2} + d_1 t^{n-2} \int_1^t \phi_0(s) g_0 \left( \frac{|u(s)|}{s^{n-1}} \right) \, ds.$$

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Considering that
\[ |u(t)| - |u(1)| \leq \int_1^t |u'(s)| \, ds, \]
we have
\[ |u(t)| \leq c'_1 t^{n-1} + c''_1 + d_0 \int_1^t \left( t^{n-1} - s^{n-1} \right) \varphi_0(s) \frac{|u(s)|}{s^{n-1}} \, ds, \]
where \( c'_1 = c_1/(n-1) \), \( c''_1 = |u(1)| - c_1/(n-1) \), \( d_0 = d_1/(n-1) \). Let \( c_0 = c'_1 + c''_1 \). Then
\[ \frac{|u(t)|}{t^{n-1}} \leq c_0 + d_0 \int_1^t \varphi_0(s) \frac{|u(s)|}{s^{n-1}} \, ds \]
for \( t \geq 1 \). Using the Lemma we get
\[ |u(t)|/t^{n-1} \leq G_0^{-1} \left( G_0(c_0) + d_0 \int_1^t \phi_0(s) \, ds \right) \leq G_0^{-1} \left( G_0(c_0) + d_0 \int_1^\infty \phi_0(s) \, ds \right) \triangleq b_0. \]
Thus (9) is true for \( i = 0 \). Suppose (9) is true for \( i = k < n - 1 \). From (8), letting \( r = n - k - 2 \), we have
\[ \frac{|u(k+2)(t)|}{t^{n-k-3}} \leq c_{k+2} + d_{k+2} \int_1^t \sum_{i=0}^k \phi_i(s) \frac{g_i(b_i)}{s^{n-k-3}} \, ds \]
\[ + \frac{d_{k+2}}{s^{n-k-2}} \int_1^t \phi_{k+1}(s) g_{k+1} \left( \frac{|u(k+1)(s)|}{s^{n-k-2}} \right) \, ds, \]
where \( c'_{k+2} = c_{k+2} + d_{k+2} \int_1^\infty \sum_{i=0}^k \phi_i(s) g_i(b_i) \, ds \). Hence
\[ |u(k+2)(t)| \leq c'_{k+2} t^{n-k-3} + d_{k+2} t^{n-k-3} \int_1^t \phi_{k+1}(s) g_{k+1} \left( \frac{|u(k+1)(s)|}{s^{n-k-2}} \right) \, ds. \]
Considering that
\[ |u(k+1)(t)| - |u(k+1)(1)| \leq \int_1^t |u(k+2)(s)| \, ds, \]
we have
\[ |u(k+1)(t)| \leq c''_{k+2} t^{n-k-2} \]
\[ + c''_{k+2} + d_{k+1} \int_1^t \left( t^{n-k-2} - s^{n-k-2} \right) \phi_{k+1}(s) g_{k+1} \left( \frac{|u(k+1)(s)|}{s^{n-k-2}} \right) \, ds, \]
where \( c''_{k+2} = c_{k+2}/(n-k-2) \), \( c'''_{k+2} = |u(k+1)(1)| - c'_{k+2}/(n-k-2) \), \( d_{k+1} = d_{k+2}/(n-k-2) \). Letting \( c_{k+1} = c''_{k+2} + |c'''_{k+2}| \), then
\[ \frac{|u(k+1)(t)|}{t^{n-k-2}} \leq c_{k+1} + d_{k+1} \int_1^t \phi_{k+1}(s) g_{k+1} \left( \frac{|u(k+1)(s)|}{s^{n-k-2}} \right) \, ds. \]
for $t \geq 1$. Using the Lemma we get
\[
|u^{(k+1)}(t)|/t^{n-k-2} \leq G_{k+1}^{-1} \left( G_{k+1}(c_{k+1}) + d_{k+1} \int_1^t \phi_{k+1}(s) \, ds \right)
\leq G_{k+1}^{-1} \left( G_{k+1}(c_{k+1}) + d_{k+1} \int_1^\infty \phi_{k+1}(s) \, ds \right) \leq b_{k+1}.
\]
Thus (9) is true for $i = k + 1$.

(iii) From equation (2) we obtain that $i = 1, 2, \ldots, n$

\[
u^{(n-i)}(t) = \sum_{j=1}^i \frac{d_{ji}}{(i-j)!} t^{i-j} - \frac{1}{(i-1)!} \int_1^t (t-s)^{i-1} f(s, u(s), u'(s), \ldots, u^{(n-1)}(s)) \, ds,
\]
where $d_{ji} = u^{(n-1)}(1)$. And from (9) we know that when $t \geq 1$,
\[
\int_1^t \left( 1 - \frac{s}{t} \right)^{i-1} |f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))| \, ds
\leq \sum_{i=0}^{n-1} \int_1^t \phi_i(s) g_i \left( \frac{|u(i)(s)|}{s^{n-i-1}} \right) \, ds
\leq \sum_{i=0}^{n-1} g_i(b_i) \int_1^\infty \phi_i(s) \, ds < \infty.
\]
Since $\int_1^t (1 - s/t)^{i-1} |f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))| \, ds$ is nondecreasing in $t$, then
\[
\lim_{t \to \infty} \int_1^t (1 - s/t)^{i-1} |f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))| \, ds
\]
exists, which follows that
\[
\lim_{t \to \infty} \int_1^t (1 - s/t)^{i-1} f(s, u(s), u'(s), \ldots, u^{(n-1)}(s)) \, ds = h_i
\]
exists. According to L’Hospital’s criterion it is easy to see that $h_1 = h_2 = \cdots = h_n \triangleq h$. Dividing both sides of (10) by $t^{i-1}$ and letting $t \to \infty$, we get
\[
\frac{u^{(n-i)}(t)}{t^{i-1}} \to \frac{d_{ii} - h_i}{(i-1)!} = \frac{u^{(n-i)}(1) - h}{(i-1)!} \triangleq a_i.
\]
(iv) Let $f(t, u_0, \ldots, u_{n-1}) > 0$ for $u_i > 0 (i = 0, \ldots, n - 1)$. For any $r > 0$, because $\int_1^\infty \phi_i(s) \, ds < \infty$, we can choose a $T$ so large that
\[
\sum_{i=0}^{n-1} g_i \left( \frac{r}{(n-i-1)!} \right) \int_T^\infty \phi_i(s) \, ds < \frac{r}{4}.
\]
If $u(t)$ is the solution of equation (2) satisfying
\[
u(T) = u'(T) = \cdots = u^{(n-2)}(T) = 0, \quad u^{(n-1)}(T) = r,
\]
then
\[
u^{(n-i)}(t) = \frac{r}{(i-1)!} (t - T)^{i-1} - \frac{1}{(i-1)!} \int_T^t (t-s)^{i-1} f(s, u(s), \ldots, u^{(n-1)}(s)) \, ds,
\]
which follows for \( t > T \)
\[
u^{(n-i)}(t)/t^{i-1} \leq r/(i-1)!, \quad i = 1, 2, \ldots, n,
\]
i.e.,
\[
u^{(i)}(t)/t^{n-i-1} \leq r/(n-i-1)!, \quad i = 0, 1, \ldots, n-1,
\]
as long as \( u^{(i)}(s) > 0 \) \( (i = 0, 1, \ldots, n-1) \) for \( s \in (T, t) \). Thus there exists a maximal interval \((T, \bar{T})\) in which
\[
0 < u^{(i)}(t)/t^{n-i-1} \leq r/(n-i-1)!, \quad \text{and} \quad u^{(n-1)}(t) > r/2.
\]
However it follows \( \bar{T} = \infty \). For otherwise
\[
u^{(n-1)}(\bar{T}) = r - \int_{T}^{\bar{T}} f(s, u(s), \ldots, u^{(n-1)}(s)) \, ds
\]
\[
\geq r - \sum_{i=0}^{n-1} \int_{T}^{\bar{T}} \phi_i(s) g_i \left( \frac{|u^{(i)}(s)|}{s^{n-i-1}} \right) \, ds
\]
\[
\geq r - \sum_{i=0}^{n-1} g_i \left( \frac{r}{(n-i-1)!} \right) \int_{T}^{\bar{T}} \phi_i(s) \, ds \geq \frac{3}{4} r.
\]
It would contradict the maximal property of \( \bar{T} \). This means \( u^{(n-1)}(t) > r/2 \) for \( T \leq t < \infty \). Hence \( a_1 \geq r/2 > 0 \). From (11) we know \( a_i > 0 \) \( (i = 1, 2, \ldots, n) \).

Let \( f(t, u_0, \ldots, u_{n-1}) < 0 \) for \( u_i > 0 \) \( (i = 0, \ldots, n-1) \). Then we can conclude that \( u^{(n-i)}(t) > 0 \) \( (i = 1, 2, \ldots, n) \) for \( t > T \). For otherwise there would be a \( \bar{T} > T \) such that \( u^{(n-i)}(t) > 0 \) for \( t \in (T, \bar{T}) \) and \( u^{(n-k)}(\bar{T}) = 0 \) for some \( k \). In view of (13) we have
\[
u^{(n-k)}(\bar{T}) = \frac{r}{(k-1)!} (\bar{T} - T)^{k-1}
\]
\[
- \frac{1}{(k-1)!} \int_{T}^{\bar{T}} (\bar{T} - S)^{k-1} f(s, u(s), \ldots, u^{(n-1)}(s)) \, ds > 0,
\]
which contradicts the assumption. Thus from (13) we know
\[
u^{(n-i)}(t)/t^{i-1} \geq \frac{r}{(i-1)!} \left( 1 - \left( 1 - \frac{T}{t} \right) \right)^{i-1} \quad \text{for} \quad t \geq T.
\]
This shows \( a_i \geq r/2(i-1)! > 0 \).

The proof is complete.

EXAMPLE. Consider the equation
\[
u'' - (2/t^3)(|uu'|)^{1/2} = 0.
\]
We have \( f(t, u, u') = -(2/t^3)(|uu'|)^{1/2} \). Hence
\[
|f(t, u, u')| \leq (1/t^2)(|u|/t + |u'|)
\]
for \( t \geq 1 \). Let \( \phi_i(t) = 1/t^2 \), \( g_i(u) = u \) \( (i = 0, 1) \). The conditions of the Theorem are satisfied. By using the theorem, we see, for every solution \( u(t) \) of equation (14)
\[
u'(t) \to a_1, \quad u(t)/t \to a_2 = a_1 \quad \text{as} \quad t \to \infty.
\]
And equation (13) has solutions such that \( a_1 > 0 \).

The conclusion cannot be drawn from [5] and other literatures.

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REFERENCES


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