ESTIMATES FOR GREEN'S FUNCTIONS

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ABSTRACT. Let \( l_q = -\nabla^2 + q(x), \) \( x \in \mathbb{R}^3, \) \( 0 \leq q \leq c(1 + |x|)^{-a}, \) \( a > 2, \)
\( l_q G_q(x,y) = \delta(x-y). \) If \( q \geq p \geq 0, q \neq p, \) then \( c|x-y|^{-1} < G_q(x,y) < G_p(x,y) \leq (4\pi|x-y|)^{-1}, x \neq y, \) for some positive \( c = c(q). \) If \( p \neq 0 \) then \( G_p < (4\pi|x-y|)^{-1}, x \neq y. \)

Introduction. If \( A \) and \( B \) are linear selfadjoint operators on a Hilbert space \( H, \)
\( (*) \quad A \geq B > 0, \) \( D(A) \subset D(B), \) then \( 0 < A^{-1} \leq B^{-1}. \)
Here \( A \geq B \) means that \( (Af,f) \geq (Bf,f) \) for all \( f \in D(A). \) However, if \( A^{-1} \) and \( B^{-1} \) are integral operators, the inequality (*) does not imply that
\( (**) \quad G_B(x,y) \geq G_A(x,y) \) for all \( x \) and \( y, \)
where \( G_A(x,y) \) is the kernel of \( A^{-1}. \) This is well known and can be easily seen already in the example when \( A^{-1} \) and \( B^{-1} \) are matrices (e.g. if
\[
A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B^{-1} = \begin{pmatrix} 5 & -1 \\ -1 & 3 \end{pmatrix},
\]
then \( B^{-1} \geq A^{-1} \) but the inequality \( B^{-1}_{jm} \geq A^{-1}_{jm} \) does not hold, where \( B^{-1}_{jm} \) are the entries of the matrix \( B^{-1}. \)

It is therefore of interest to give conditions under which (**) holds. For matrices such conditions are known, the corresponding matrices are called \( M \)-matrices (see [1, 2]). For positive definite elliptic differential operators of second order in \( L^2(D), \) where \( D \) is a bounded domain, it is established in [3] that their Green’s functions are nonnegative. See also a discussion in [7, p. 209] of Beurling-Deny criteria for a selfadjoint operator to generate a positivity preserving semigroup. For higher order elliptic positive definite operators (such as biharmonic operator, for example) Green’s function is not necessarily pointwise positive (some references can be found in [3]).

Formulation of the results. Assume that \( A \) is selfadjoint in \( L^2 = L^2(\mathbb{R}^3) \)
operator defined by the expression \( Au = -\nabla^2 u + q(x)u \) on \( D(A) = H^2(\mathbb{R}^3), \) where \( H^2(\mathbb{R}^3) \) is the standard Sobolev space, and
\( (1) \quad 0 \leq q(x) \leq c(1 + |x|)^{-a}, \quad a > 2. \)
By \( c \) we denote various positive constants, \( |x| \) is the length of the vector \( x \in \mathbb{R}^3. \) Let
\( (2) \quad 0 \leq p(x) \leq q(x) \)

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and let $G_q(x,y)$ denote Green's function:

\[(3)\quad [-\nabla^2 + q(x)]G_q(x,y) = \delta(x-y).\]

**Lemma 1.** If (1) and (2) hold then

\[(4)\quad (4\pi|x-y|)^{-1} \geq G_p(x,y) \geq G_q(x,y) > 0 \quad \text{for all } x,y \in \mathbb{R}^3.\]

If $p \neq 0$ and $q \neq p$, then

\[(4')\quad (4\pi|x-y|)^{-1} > G_p(x,y) > G_q(x,y) > 0, \quad \text{for } x \neq y.\]

**Remark 1.** Estimates of the type (4) are useful in applications [5, pp. 309-317, 6].

**Proof.** First let us prove the right inequality (4) using the strong maximum principle [4]. Suppose that $G_q(x_0,y_0) := G(x_0,y_0) < 0$ for some $x_0$ and $y_0$. Clearly $x_0 \neq y_0$ since $G(x,y) \to +\infty$ as $x \to y$. Since $G(x,y) \to 0$ as $|x-y| \to \infty$, one concludes that the function $G(x,y_0) := u(x)$ attains a nonpositive minimum at a point $\xi \neq y_0$. Consider the function $u(x) := G(x,y_0)$ in a ball $B_a = \{x: |\xi-x| \leq a\}$ which does not contain $y_0$. This function solves the equation

\[(5)\quad \nabla^2 u(x) - q(x)u(x) = 0 \quad \text{in } B_a.\]

Since $q(x) \geq 0$, we conclude that $u$ cannot attain in $B_a$ a nonpositive minimum at the point $\xi$ unless $u \equiv \text{const}$ for $x \neq y_0$. The function $u(x) = G(x,y_0) \equiv \text{const}$: if $u \equiv \text{const}$ then the property $G(x,y_0) \to \infty$ as $x \to y_0$ cannot hold. This contradiction proves that $G_q(x,y) > 0$ for all $x,y$.

The left inequality (4) follows from the equation

\[(6)\quad G_p(x,y) = g(x,y) - \int_{\mathbb{R}^3} g(x,z)p(z)G_p(z,y)\,dz, \quad g := (4\pi|x-y|)^{-1}.\]

Since $p \geq 0$ and $g \geq 0$ one concludes that $G_p \leq g$.

In order to prove that $G_p \geq G_q$, define

\[(7)\quad v := G_p(x,y) - G_q(x,y).\]

Subtract from equation (3) the equation for $G_p$ to get

\[(8)\quad \nabla^2 v - pv = (p-q)G_q \leq 0\]

where assumption (2) was used. Since $v \to 0$ as $|x-y| \to \infty$ and $v$ is bounded on compact sets, the strong maximum principle says that $v$ cannot attain a nonpositive minimum, therefore, $v > 0$. Thus $G_p > G_q$.

Note that in order that Green's function $G_q(x,y)$ in Lemma 1 be positive, it is not necessary that $q(x)$ be nonnegative. Indeed, the following lemma holds. Let $a(x) \geq 0$, $a_M(r) = \sup_{|x| = r} a(x)$, $a(\infty) = 0$, $a \in L^1(\mathbb{R}^3)$, $\gamma > \frac{3}{2}$.

**Lemma 2.** If for some $u_0(x) \geq c(|x|+1)^{-\alpha}$, $0 < \alpha < 1$, $c = \text{const} > 0$, the equation $(-\Delta - a(x))u_0 = 0$ holds in $\mathbb{R}^3$, then $G_q(x,y) > 0$ in $\mathbb{R}^3$ provided that $q(x) \geq -a(x)$. If $0 \geq q \geq -a(x)$ and $q(x) \neq 0$ then $G_q(x,y) > (4\pi|x-y|)^{-1}, x \neq y$.

**Proof.** Define $V$ by the equation $G_q(x,y) = u_0(x)V(x,y)$. Then

\[\Delta_x G_q(x,y) = V\Delta_x u_0 + 2\nabla_x u_0 \cdot \nabla_x V + u_0 \Delta_x V(x,y).\]
Substitute this into the equation $-\Delta G_q + q G_q = \delta(x - y)$ to get for fixed $y$

$$-u_0 \Delta V - 2 \nabla u_0 \cdot \nabla V + (-\Delta u_0 + q u_0) V = \delta(x - y).$$

Note that $V$ is by definition positive in a neighborhood of $y$. Therefore it is sufficient to prove that $V > 0$ outside of this neighborhood. This is done by proving that $V$ cannot attain a nonpositive value outside of the above neighborhood. If it does, then the maximum principle says that $V \equiv \text{const}$ and this is a contradiction.

Since in (9) we have $-\Delta u_0 + q u_0 \geq -\Delta u_0 - a(x) u_0 = 0$, and $V = u_0^{-1} G_q = O(|x|^{a-1})$ as $|x| \to \infty$, so that $|V| \to 0$ as $|x| \to \infty$, the maximum principle applies to $V$. Hence $V(x, y) > 0$ and this yields positivity of $G_q(x, y)$. To prove the last statement of Lemma 2, start with the formula $G_q = g - \int_{R^3} q G_q \, dz$, which is analogous to (6), and note that the integral term is positive since $q \leq 0$, $q \neq 0$, and $G_q > 0$, as we have already proved.

**Example.** Let $|x| = r$, $u_0 := (r + \varepsilon)^{-\alpha}$, $0 < \alpha < 1$, $\varepsilon > 0$ is an arbitrary small number, $a(r) = (r + \varepsilon)^{-2}[\alpha(1 - \alpha) + 2 \varepsilon \alpha r^{-1}]$. If $q \geq -a(r)$ then, by Lemma 2, $G_q(x, y) > 0$. If $\alpha = \frac{1}{2}$ and $q \geq -(r + \varepsilon)^{-2}[\frac{1}{4} + \varepsilon r^{-1}]$ then $G_q > 0$. In particular, if $q \geq -(r + \varepsilon)^{-2}/4$ then $G_q > 0$. This result is close to the best possible as follows from Remark 2.

**Remark 2.** Let $q_m(r) := \inf_{|x| = r} q(x)$, $q_M(r) := \sup_{|x| = r} q(x)$. The following result is known [9, §48]. If $x \in R^3$ and

$$\lim_{r \to \infty} \inf r^2 q_m(r) > -1/4$$

then the equation $-\Delta u + qu = 0$ does not have nontrivial solutions with nodal surface in a neighborhood of infinity, that is there is no nodal surface in the region $\Omega_R := \{x : |x| > R\}$ for some $R > 0$, (in this case the equation $-\Delta u + qu = 0$ is called nonoscillatory). If

$$\lim_{r \to \infty} \sup r^2 q_M < -1/4$$

then the equation $-\Delta u + q(x) = 0$ is oscillatory, that is any solution to this equation has nodal surfaces in any neighborhood $\Omega_R$ of infinity (i.e. for any $R > 0$ sufficiently large).

If the equation $(-\Delta - a(x)) u = 0$ has a positive solution $u_0$, then this equation is not oscillatory. Since $(-a)_m = -a_M$ one has for $q = -a(x)$ the sufficient condition

$$\lim_{r \to \infty} \inf r^2 a_M(r) < 1/4,$$

for the equation $[-\Delta - a(x)] u = 0$ to be nonoscillatory. This condition generalizes Kneser’s condition known in the one dimensional case as a sufficient condition for the equation $-u'' - a(x) u = 0$, $x \in R^1$, to be nonoscillatory. Practically one cannot take $a(x)$ large: if $a(x)$ were large, then the operator $-\nabla^2 - a(x)$ would have a negative eigenvalue $[-\nabla^2 - a(x)] \psi = -\lambda \psi$, $\lambda > 0$, where $\psi$ decays exponentially at infinity. The corresponding eigenfunction is strictly positive, and the equation $[-\Delta - a(x)] u = 0$ cannot have positive solution which grows not faster than a power of $|x|$ at infinity. Indeed, if $[-\Delta - a(x)] \psi = -\lambda \psi$, $\psi > 0$, and $[-\Delta - a(x)] w = 0$, $w > 0$, then

$$-\lambda \int_{R^3} \psi w \, dx = \int_{R^3} w (-\Delta - a) \psi \, dx = \int_{R^3} [-\Delta - a(x)] w \psi \, dx = 0.$$
This is a contradiction since \( \int \psi u \, dx > 0 \). An integration by part was used and the integrals over the large sphere go to zero as the radius of the sphere grows since \( \psi \) decays exponentially at infinity.

Therefore Lemma 2 is useful practically in the case when \( a(x) \) is a function for which the operator \(-\nabla^2 - a(x)\) does not have negative eigenvalues and for which zero is a resonance with positive resonance function, that is with positive solution to the equation \((-\nabla^2 - a(x))w = 0\).

The following theorem gives a positive lower bound of Green's function in \( R^3 \).

**THEOREM 1.** Under the assumption in Lemma 1, the estimates

\[
(4\pi|x-y|)_1^{-1} \geq G_p(x,y) \geq G_q(x,y) > c|x-y|_1^{-1}, \quad x \neq y,
\]

hold where \( c = c_q > 0 \), and the inequalities are strict if \( p \neq q, p \neq 0 \).

**PROOF.** Lemma 1 shows that the first two inequalities hold. The last inequality is proved in [6, p. 1341, formula (4)].

**Extensions.** In this section we extend the results in several directions.

(1) The result in Theorem 1 and its proof remain valid in \( R^d \) for \( d \geq 3 \) after the obvious modifications: \((4\pi|x-y|)_1^{-1}\) should be substituted by \(\Gamma(d/2)|x-y|^{2-d}/2(d-2)^{d/2}\) and \(|x-y|_1^{-1}\) by \(|x-y|^{2-d}\).

(2) General second order selfadjoint elliptic operators

\[
Lu = - \sum_{j,m=1}^d \partial_j (a_{mj}(x) \partial_m u) + q(x)u,
\]

where

\[
\lambda \sum_{j=1}^d |t_j|^2 \leq \sum_{j,m=1}^d a_{mj}(x) t_m t_j \leq \Lambda \sum_{j=1}^d |t_j|^2.
\]

\( \lambda \) and \( \Lambda \) are positive constants which do not depend on \( x \), \( a_{mj}(x) \in C^1 \) and \( q(x) \) satisfies assumption (1), can be studied in place of \(-\Delta + q(x)\).

(3) In this section we discuss the Green functions with boundary conditions on \( \partial D \), where \( D \subset R^d, d \geq 3 \), \( D \) is a domain with smooth boundary \( \partial D \), \( \overline{D} = D \cup \partial D \). By \( B_D u = 0 \), \( B_N u = 0 \) and \( B_R u = 0 \) we mean \( u|_{\partial D} = 0 \), \( \partial u/\partial n|_{\partial D} = 0 \) and \( \partial u/\partial n + \sigma(x) u|_{\partial D} = 0 \) respectively, where \( 0 \leq \sigma(x) \) is smooth, \( n \) is the outer normal. By \( P_D, P_N, P_R \) we denote the set of functions \( a(x) \in C(\overline{D}) \) for which there exists a \( u(x) \geq 0 \) such that \((-\Delta + a(x))u(x) \geq 0 \) in \( D \) and \( u \) satisfies \( B_D u = 0 \), \( B_N u = 0 \) or \( B_R u = 0 \) respectively. Let \( \lambda_1(a) \) denote the first eigenvalue of \(-\Delta + a(x)\) and \( u_0(x) \) denote the nonnegative eigenfunction corresponding to \( \lambda_1(a) \) under one of the three boundary conditions. We use \( P \) to denote one of the sets \( P_D, P_N \) or \( P_R \) in what follows. It is easy to see that \( a(x) \in P \) if and only if \( \lambda_1(a) \geq 0 \).

**REMARK 3.** If \( a(x) \geq 0 \) then \( u_0(x)|_{\partial D} > 0 \) under Neumann or Robin boundary condition (\( B_N u = 0 \) or \( B_R u = 0 \)).

**PROOF.** From the monotonicity of the dependence of first eigenvalue \( \lambda_1(a) \) on \( a(x) \), one concludes that \( a(x) \in P \). Since \( u_0(x) > 0 \) in \( D \), \( u_0(x)|_{\partial D} \geq 0 \). Suppose \( u_0(s) = 0 \) for some \( s \in \partial D \). Since \(-\Delta u_0 + au_0 = \lambda_1(a) u_0 \geq 0 \), the strong maximum principle and Hopf's lemma imply that \( \partial u_0(s)/\partial n < 0 \). This is a contradiction to the assumption \( B_N u = 0 \) or \( B_R u = 0 \).
Lemma 3. Let \( a(x) \geq 0 \) and continuous function \( q(x) \geq a(x) - \lambda_1(a) \), then \( q(x) \in P \). If \( u(x) > 0 \) for all \( x \in D \) satisfies the inequality \((-\Delta + q)u(x) \geq 0\) and one of the boundary conditions \( B_N u = 0 \) or \( B_R u = 0 \), then \( u(x)|_{\partial D} > 0 \).

Proof. Since the first eigenvalue \( \lambda_1(q) \) of \(-\Delta + q(x)\) is not less than that of the operator \(-\Delta + (a(x) - \lambda_1(a))\) which is \( \lambda_1(a) - \lambda_1(a) = 0 \), it follows that \( q(x) \in P \).

Let \( u_0(x) \) denote the (nonnegative) eigenfunction corresponding to \( \lambda_1(a) \) with one of the conditions \( B_N u = 0 \) or \( B_R u = 0 \). Define the function \( V(x) \) by the equation \( u(x) = u_0(x)V(x) \). Then

\[
0 \leq (-\Delta + q)u = -u_0\Delta V - 2V \cdot \nabla u_0 + (q + \lambda_1(a) - a(x))u_0V.
\]

Note that \( q + \lambda_1(a) - a(x) \geq 0 \) by the assumption and that on boundary \( \partial D \) we have:

(i) In the case of Neumann condition \( B_N u = 0 \),

\[
0 = \frac{\partial u}{\partial n} = V \frac{\partial u_0}{\partial n} + u_0 \frac{\partial V}{\partial n} = u_0 \frac{\partial V}{\partial n}.
\]

One thus concludes by Remark 3 that \( \partial V / \partial n = 0 \).

(ii) In the case of Robin condition

\[
0 = \frac{\partial u}{\partial n} + \sigma u = V \left( \frac{\partial u_0}{\partial n} + \sigma u_0 \right) + u_0 \frac{\partial V}{\partial n} = u_0 \frac{\partial V}{\partial n}.
\]

One can also conclude that \( \partial V / \partial n = 0 \).

By the strong maximum principle, \( V \) either attains its nonpositive minimum on \( \partial D \) or it is a constant. Using Hopf's lemma, which says that at the point \( s \in \partial D \) of minimum \( \partial V(s)/\partial n < 0 \), and the result \( \partial V / \partial n = 0 \) one concludes that \( V \) cannot attain its minimum on \( \partial D \) and therefore \( V \equiv \text{constant} \). Since \( u_0(x) > 0 \) in \( D \), it follows that this \( V \equiv \text{const} > 0 \), and, by Remark 3, that \( u(x) = u_0(x)V > 0 \) for \( x \in \partial D \).

We now discuss the estimates for Green functions under the Dirichlet, Neumann or Robin boundary conditions. By \( G_{qD}, G_{qR}, \) and \( G_{qN} \) we denote the corresponding Green functions. In the case of Neumann condition we always assume \( q(x) \neq 0 \) so that the Green function \( G_{qN} \) is uniquely determined.

We then have the following propositions.

Proposition 1. If \( q(x) \geq a(x) - \lambda_1(a) \) and \( a(x) \geq 0 \) then \( G_q(x, y) > 0 \) for \( x, y \in D \).

Let \( a(x) \equiv 0 \) and \( \lambda_1 \) be the first eigenvalue of the Laplacian \(-\Delta\) under one of the three boundary conditions, Proposition 1 implies

Corollary. \( G_q(x, y) > 0 \) for \( x, y \in D \), provided that \( q(x) \geq -\lambda_1 \).

Proof of Proposition 1. Since \( q(x) \in P \) by Lemma 3, it follows that \(-\Delta + q(x)\) is nonnegative definite. Therefore the positivity of \( G_q \) follows from the Aronszajn-Smith theorem (see §5 in [3]).

One can also prove Proposition 1 by applying the strong maximum principle to the function \( V(x, y) \), where \( V(x, y) \) is defined by \( G_q(x, y) = u_0(x)V(x, y) \), and using the argument in Lemma 3.
PROPOSITION 2. If \( q_1(x) \geq q_2(x) \geq a(x) - \lambda_1(a) \) for some \( a(x) \geq 0 \), and \( q_1 \neq q_2 \), then \( G_{q_2}(x, y) > G_{q_1}(x, y) > 0 \) for \( x \neq y, \ x, y \in D \).

PROOF. Strict positivity of \( G_{q_i} \) is a result of Proposition 1. The first inequality follows from the equation

\[
G_{q_2}(x, y) - G_{q_1}(x, y) = \int_D (q_1(z) - q_2(z))G_{q_1}(x, z)G_{q_2}(z, y)\, dz
\]

and from the positivity of \( G_{q_i}, \ i = 1, 2 \).

PROPOSITION 3. If \( q(x) > a(x) - \lambda_1(a) \) for some \( 0 < a(x) \in P \), then

\[
G_{q_D}(x, y) < G_{q_R}(x, y) < G_{q_N}(x, y) \quad \text{for} \ x, y \in \overline{D}, \ x \neq y,
\]

and

\[
0 < G_{q_D}(x, y) \quad \text{for} \ x, y \in D.
\]

PROOF. Step 1. Let us prove the first inequality in (10). The argument in the proof of Lemma 3 shows that \( G_{q_R}(x, y) > 0 \) for \( x \) or \( y \) on \( \partial D \). Thus \( (-\Delta + q)(G_{q_R} - G_{q_D}) = 0, (G_{q_R} - G_{q_D})|_{\partial D} > 0 \). Define \( V \) by the equation

\[
G_{q_R}(x, y) - G_{q_D}(x, y) = u_0(x)V(x, y),
\]

where \( u_0(x) \) is the (positive) eigenfunction of \( -\Delta + a(x) \) corresponding to the eigenvalue \( \lambda_1(a) \) under Robin condition. One has

\[
-q \lambda_1(a) - a(x) \geq 0 \quad \text{and} \quad V(x, y) > 0 \quad \text{for} \ x \text{ or } y \text{ on } \partial D.
\]

Note that \( G_{q_R}(x, y) - G_{q_D}(x, y) = G_{q_R}(x, y) > 0 \) for \( x \) or \( y \) on \( \partial D \) and since \( u_0 > 0 \) on \( \partial D \), formula (12) implies that \( V(x, y) > 0 \) for \( x \) or \( y \) on \( \partial D \). The maximum principle implies that \( V(x, y) > 0 \) for \( x, y \in \overline{D} \). Therefore

\[
G_{q_R}(x, y) > G_{q_D}(x, y) \quad \text{for} \ x, y \in \overline{D}.
\]

Step 2. Let us prove the second inequality in (10). Note that

\[
\frac{\partial}{\partial n} (G_{q_R} - G_{q_N})|_{\partial D} = \frac{\partial}{\partial n} G_{q_R}|_{\partial D} = -\sigma(x)G_{q_R} \leq 0.
\]

Define \( V \) by the equation

\[
G_{q_R}(x, y) - G_{q_N}(x, y) = u_0(x)V(x, y)
\]

where \( u_0(x) \) is the (positive) eigenfunction of the operator \( -\Delta + a(x) \) corresponding to Neumann boundary condition with first eigenvalue \( \lambda_1(a) \). Then

\[
\frac{\partial(u_0V)}{\partial n}|_{\partial D} = \frac{\partial(G_{q_R} - G_{q_N})}{\partial N}|_{\partial D} \leq 0.
\]

Thus \( u_0(\partial_x V/\partial n)|_{\partial D} \leq 0 \). Since \( u_0(x)|_{\partial D} > 0 \) by Remark 3, it follows that

\[
\frac{\partial_x V}{\partial n}|_{\partial D} \leq 0.
\]

Equation (13) and the strong maximum principle imply that \( V(x, y) \) can attain its nonnegative maximum only at some \( x \) on \( \partial D \) for any given \( y \in \overline{D} \). By
Hopf's lemma, $\partial_x V/\partial n|_{\partial D} > 0$. This contradicts to (16). Therefore $G_q R(x, y) - G_q N(x, y) < 0$ for $x, y \in \overline{D}$, $x \neq y$.

Note that there is no positive lower bound for Green's function $G_q(x, y)$ which is uniform in $q$. Indeed if $q(x) \to +\infty$ in $\overline{D} \subset D$, then $G_q(x, y) \to 0$ in $\overline{D}$ as follows from the results in [8].

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