BEST UNIFORM APPROXIMATION
BY BOUNDED ANALYTIC FUNCTIONS

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ABSTRACT. This paper gives a counterexample to the conjecture that the continuity of the conjugate $\hat{f}$ of an $f \in C(T)$ implies the continuity of the best uniform approximation $g \in H^\infty(T)$ of $f$. It also states two conditions which imply the continuity of $g$.

Let $L^\infty(T)$ the space of bounded measurable functions on the unit circle $T$, $H^\infty(T)$ the subalgebra of $L^\infty(T)$ consisting of nontangential limits of bounded analytic functions in the unit disk and write $\|f\|_\infty$ for the (essential supremum) norm of $f \in L^\infty(T)$. Also, let $C(T)$ be the space of all continuous functions on $T$.

It is known that any $f \in L^\infty(T)$ has at least one best approximation $g \in H^\infty(T)$, in the sense that

$$d = \|f - g\|_\infty = \inf_{h \in H^\infty} \|f - h\|_\infty$$

and that, by duality

$$d = \sup \left\{ \left| \int_0^{2\pi} f(\theta)F(\theta) \frac{d\theta}{2\pi} \right| : F \in H^1(T), F(0) = 0, \|F\|_1 \leq 1 \right\}$$

where $H^p(T) (0 < p < \infty)$ is the Hardy space of all nontangential limits of functions $F$ analytic in the unit disc such that

$$\|F\|_p = \sup_{0 < r < 1} \left| \int_0^{2\pi} F(re^{i\theta}) \frac{d\theta}{2\pi} \right| < +\infty.$$

Moreover, if $f$ is continuous, then the best approximation $g$ of $f$ is unique and there is at least one $F$, for which the supremum (*) is attained. Also $f, g$ and any of those maximizing $F$'s are connected by

$$f(\theta) - g(\theta) = \|f - g\|_\infty \frac{F(\theta)}{|F(\theta)|} \quad \text{a.e.} \ (d\theta)$$

which implies

$$|f(\theta) - g(\theta)| = \|f - g\|_\infty = d \quad \text{a.e.} \ (d\theta).$$

We need the following result (see [1 or 2]):

**THEOREM 1 (CARLESON-JACOBS).** If $f \in C(T), g \in H^\infty(T), F \in H^1(T)$ are connected by (1), then

(a) $F \in H^p(T)$, for all $p < +\infty$,
(b) if $\tau \in [0,2\pi]$ and if

$$f_\tau(\theta) = f(\theta) - f(\tau), \quad g_\tau(\theta) = g(\theta) - f(\tau)$$

then there is $\delta > 0$ and $r_0 > 0$ such that

$$|g_\tau(z)| \geq \frac{1}{2} \cdot \|f - g\|_\infty \quad \text{on} \quad W_\tau = \{z = re^{i\theta} : |\theta - \tau| < \delta, r_0 < r < 1\}$$

where $\delta$ and $r_0$ can be independent of $\tau$.

We consider the problem of how the regularity of $f$ affects the regularity of $g$. In [1] the following is proved.

**Theorem 2.** If $f$ is Dini-continuous, i.e. if $\int_0^\infty (\omega(t)/t) \, dt < +\infty$, where $\omega(t) = \sup_{|x - y| \leq t} |f(x) - f(y)|$ is the modulus of continuity of $f$, then its best approximation $g$ is also continuous.

In [1] a function $f$ is constructed, continuous but not Dini-continuous, whose best approximation $g$ is not continuous.

Because the Dini-continuity of $f$ implies the continuity of its conjugate $\hat{f}$ and because of the proof in [1], it was conjectured that, for $f \in C(T)$, the continuity of $\hat{f}$ and the continuity of $g$ are equivalent.

It was proved by Sarason that the continuity of $g$ does not imply the continuity of $\hat{f}$. See [2, p. 177].

This paper provides a counterexample for the other half of the conjecture. It constructs a continuous function $f$, whose conjugate $\hat{f}$ is continuous, but whose best approximation $g$ is not. We also give two further conditions on $f$ which imply $g$ is continuous.

In the following $\overline{f}$ is the complex conjugate of $f$.

**Theorem 3.** If $f \in A(T) = H^\infty(T) \cap C(T)$ and $\int_0^\infty (\omega^2(t)/t) \, dt < +\infty$, then $g$, the best approximation of $f$, is continuous.

**Theorem 4.** If $f \in A(T)$ and $|f|^2 \in C(T)$ and $\int_0^\infty (\omega^3(t)/t) \, dt < +\infty$, then $g$ is continuous.

**Theorem 5.** There exists a function $f$, such that $\overline{f} \in A(T)$, but such that its best approximation $g$ is not continuous.

Since $\overline{f} = -if$ when $f \in A(T)$, the function in Theorem 5 has a continuous conjugate.

**Proof of Theorem 3.** Suppose $\|f - g\|_\infty = 1$. Fix $\tau \in [0,2\pi]$. Then, from Theorem 1(b), $g_\tau(z)$ has a well-defined logarithm on $W_\tau$, which is given by

$$\log g_\tau(z) = \frac{1}{2\pi} \int |\theta - \tau| \leq \delta \log |g_\tau(\theta)| \cdot \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta + R_\tau(z), \quad z \in W_\tau,$$

where $R_\tau(z)$ is the integral over $|\theta - \tau| > \delta$ plus the logarithm of the inner factor of $g_\tau$. Since $|g_\tau| \geq \frac{1}{2}$ on $W_\tau$, this inner factor is analytic across $|\theta - \tau| < \delta$. So $R_\tau(z)$ and its derivative are bounded on $|z - e^{i\tau}| < \delta_1$, for some $\delta_1 < \delta$, independent of $\tau$. This implies

$$|R_\tau(z) - R_\tau(w)| \leq c|z - w| \quad \text{for} \quad |z - e^{i\tau}| < \delta_1, \quad |w - e^{i\tau}| < \delta_1.$$
We also have

\[ |f(\theta) - g(\theta)| = 1 \quad \text{a.e.} \quad (d\theta) \]
from which

\[ |f_r(\theta) - g_r(\theta)| = 1 \quad \text{a.e.} \quad (d\theta) \]
and

\[ |g_r|^2 = 1 + 2 \cdot \text{Re}(\overline{f_r} \cdot g_r) - |f_r|^2. \]

Therefore

\[ \log |g_r| = \frac{1}{2} \log |g_r|^2 = \frac{1}{2} \left[ 2 \cdot \text{Re}(\overline{f_r} \cdot g_r) - |f_r|^2 + O(|f_r|^2) \right] = \text{Re}(\overline{f_r} g_r) + O(|f_r|^2), \]

and

\[ \log g_r(z) = \frac{1}{2\pi} \int_{|\theta - \tau| \leq \delta} \text{Re}(\overline{f_r} g_r) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta + \frac{1}{2\pi} \int_{|\theta - \tau| > \delta} O(|f_r|^2) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta + R_r(z). \]

Since \( \overline{f_r} \) is analytic, \( \overline{f_r} g_r \) is also analytic, which implies that

\[ \frac{1}{2\pi} \int_0^{2\pi} \text{Re}(\overline{f_r} g_r) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta = \overline{f_r}(z) \cdot g_r(z). \]

Thus:

\[ \log g_r(z) - \overline{f_r}(z) g_r(z) = \frac{1}{2\pi} \int_{|\theta - \tau| \leq \delta} O(|f_r|^2) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta + R^*_r(z) \]

where

\[ R^*_r(z) = R_r(z) - \frac{1}{2\pi} \int_{|\theta - \tau| > \delta} \text{Re}(\overline{f_r} g_r) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta \]

and so, by (2),

\[ |R^*_r(z) - R^*_r(w)| \leq c|z - w| \quad \text{for} \quad |z - e^{i\tau}| < \delta_1, \quad |w - e^{i\tau}| < \delta_1. \]

If \( z \) is in a truncated cone \( \Gamma(\tau) \), which is inside \( |z - e^{i\tau}| < \delta_1 \) and has vertex \( e^{i\tau} \), then

\[ \left| \frac{e^{i\theta} + z}{e^{i\theta} - z} \right| < \frac{c}{|\theta - \tau|}, \]

and so

\[ |\log g_r(z) - \overline{f_r}(z) g_r(z) - R^*_r(z)| \leq c \int_0^\delta \frac{\omega^2(t)}{t} \, dt. \]

Since \( \overline{f_r}(z) \to 0 \) as \( z \to e^{i\tau} \),

\[ |\log g_r(z) - R^*_r(z)| \leq c \int_0^\delta \frac{\omega^2(t)}{t} \, dt + \eta(\delta) \]

where \( \eta(\delta) \to 0 \) as \( \delta \to 0 \). Hence, by (3),

\[ |g_r(z) - g_r(w)| \leq c|z - w| + \eta_1(\delta), \quad z, w \in \Gamma(\tau), \]

where \( \eta_1(\delta) \to 0 \) as \( \delta \to 0 \).
Now, if \( \sigma \) and \( \tau \) are close to each other and \( z \in \Gamma(\tau) \cap \Gamma(\sigma) \) then
\[
|g(e^{i\tau}) - g(e^{i\sigma})| \leq |g(e^{i\tau}) - g(z)| + |g(z) - g(e^{i\sigma})| \\
= |g_r(e^{i\tau}) - g_r(z)| + |g_{e\sigma}(e^{i\sigma}) - g_{e\sigma}(z)| \\
\leq c|e^{i\tau} - z| + c|e^{i\sigma} - z| + 2n_1(\delta) \leq c|\tau - \sigma| + 2n_1(\delta),
\]
and
\[
\lim_{\sigma \to \tau} |g(e^{i\tau}) - g(e^{i\sigma})| \leq 2n_1(\delta)
\]
so that
\[
\lim_{\sigma \to \tau} g(e^{i\sigma}) = g(e^{i\tau})
\]
and \( g \) is continuous.

**Proof of Theorem 4.** Now we carry the expansion of \( \log |g_r| \) one step further:
\[
\log |g_r| = \frac{1}{2} \left[ 2 \text{Re}(\overline{f_r} g_r) - |f_r|^2 - \frac{(2 \text{Re}(\overline{f_r} g_r) - |f_r|^2)^2}{2} + O(|f_r|^3) \right]
\]
\[
= \text{Re}(\overline{f_r} g_r) - \frac{1}{2} |f_r|^2 - (\text{Re}(\overline{f_r} g_r))^2 + O(|f_r|^3)
\]
\[
= \text{Re}(\overline{f_r} g_r) - \frac{1}{2} |f_r|^2 - \frac{1}{2} |\overline{f_r} g_r|^2 - \frac{1}{2} \text{Re}(\overline{f_r} g_r)^2 + O(|f_r|^3)
\]
\[
= \text{Re}(\overline{f_r} g_r) - \frac{1}{2} \text{Re}(\overline{f_r} g_r)^2 - \frac{1}{2} |f_r|^2 \\
- \frac{1}{2} |f_r|^2 (1 + 2 \text{Re}(\overline{f_r} g_r) - |f_r|^2) + O(|f_r|^3)
\]
\[
= \text{Re}(\overline{f_r} g_r) - \frac{1}{2} \text{Re}(\overline{f_r} g_r)^2 - |f_r|^2 + O(|f_r|^3).
\]
Now, because
\[
\frac{1}{2\pi} \int_0^{2\pi} \text{Re}(\overline{f_r} g_r)^2 \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta = (\overline{f_r}(z) g_r(z))^2
\]
since \( \overline{f} \in H^\infty(T) \), we get
\[
\log g_r(z) - \overline{f_r}(z) g_r(z) + \frac{1}{2} (\overline{f_r}(z) g_r(z))^2
\]
\[
= -\frac{1}{2\pi} \int_{|\theta - \tau| \leq \delta} |f_r(\theta)|^2 \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta + \frac{1}{2\pi} \int_{|\theta - \tau| \leq \delta} O(|f_r|^3) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta + R_\tau^{**}(z)
\]
where
\[
R_\tau^{**}(z) = R_\tau^*(z) + \frac{1}{2\pi} \int_{|\theta - \tau| > \delta} \text{Re}(\overline{f_r} g_r)^2 \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta
\]
and so
\[
|R_\tau^{**}(z) - R_\tau^{**}(w)| \leq c|z - w| \quad \text{for } |z - e^{i\tau}| \leq \delta_1, \quad |w - e^{i\tau}| \leq \delta_1.
\]
Now, the continuity of \( |\overline{f}|^2 \) implies the continuity of \( |f_r|^2 \), and this implies the continuity of the first integral. The rest of the proof proceeds as in Theorem 3.

**Proof of Theorem 5.** Consider the function
\[
u(t) = \begin{cases} 
-\alpha_1 \log |\log t|, & 0 < t < \frac{1}{2}, \\
-\alpha_2 \log |\log |t||, & -\frac{1}{2} < t < 0,
\end{cases}
\]
extended to be smooth in \([0, \pi] - \{0\}\), and consider the harmonic extension \( \tilde{u}(z) \) of \( u(t) \) inside the unit disk, its conjugate \( \tilde{u}(z) \) and \( f(z) = e^{u(z) - i\Re(z)} \).
Then, since \( \tilde{u}(t) \) is continuous in \([-\pi, \pi] \setminus \{0\} \), and \( |f(z)| = e^{u(z)} \to 0 \) as \( z \to 1 \), we see that \( \tilde{f} \in A(T) \).

If \( \frac{1}{3} < \alpha_1 \leq \frac{1}{2} \) and \( \frac{1}{2} < \alpha_2 \), then

\[
\int_0^{1/2} \frac{|f(t)|^3}{t} \, dt < +\infty \quad \text{and} \quad \int_{-1/2}^{0} \frac{|f(t)|^3}{|t|} \, dt < +\infty
\]

but

\[
\int_0^{1/2} \frac{|f(t)|^2}{t} \, dt = +\infty \quad \text{and} \quad \int_{-1/2}^{0} \frac{|f(t)|^2}{|t|} \, dt < +\infty.
\]

The last two imply that

\[
|f|^2(r) \to +\infty \quad \text{as} \quad r \to 1^-.
\]

From

\[
\log g(r) - \overline{f}(r)g(r) + \frac{1}{2}(\overline{f}(r) \cdot g(r))^2
= |f|^2(r) + i\overline{f}^2(r) + \frac{1}{2\pi} \int_{|\theta - \pi| \leq \delta} O(|f|^3) \frac{e^{i\theta} + r}{e^{i\theta} - r} \, d\theta + R(r)
\]

we get that

\[
\arg g(r) \to +\infty \quad \text{as} \quad r \to 1^-.
\]

Thus \( g \) is not continuous.

REFERENCES


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