CLOSED SETS WITHOUT MEASURABLE MATCHING

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ABSTRACT. We construct a rectangle in the unit square such that its perimeter contains a matching (i.e. the graph of a bijection of the unit interval onto itself) but does not contain a Borel matching or a matching measurable with respect to the linear measure.

Let \( I \) and \( I^2 \) denote the unit interval and the unit square, respectively. A subset of \( I^2 \) is called a matching if it is the graph of a one-to-one map of \( I \) onto itself. By a theorem of D. König [5], if each section \( K_x = \{ y : (x, y) \in K \} \) (\( x \in I \)) and \( K^y = \{ x : (x, y) \in K \} \) (\( y \in I \)) of the set \( K \subset I^2 \) consists of exactly \( n \) elements (\( n \) is finite), then \( K \) contains a matching. For infinite \( n \) this was proved by J. Kaniewski and C. A. Rogers [4].

It may happen that a Borel subset of \( I^2 \) contains a matching but does not contain a Borel matching or even a matching measurable with respect to the linear measure. Borel sets with this property were constructed by Kaniewski and Rogers [4] and Mauldin [6] (see also [2, §4]). The set given in [4] has countably infinite sections; the sections of Mauldin’s set are uncountable. In this note we present a closed set with the property above. It is the union of four segments and has finite sections.

**THEOREM.** Let \( u \in I \) be an irrational number and let \( R \) denote the perimeter of the rectangle with vertices \( A_0(1, 1-u), A_1(1-u, 1), A_2(0, u), \) and \( A_3(u, 0) \). Then \( R \) contains a matching but does not contain a Borel matching. Moreover, no matching in \( R \) is measurable with respect to the linear (Hausdorff) measure.

**PROOF.** By symmetry, we may assume \( 0 < u < 1/2 \). The set \( R \) defines a bipartite graph \( G \) as follows. Let \( I_1 \) and \( I_2 \) be two copies of \( I \), and let the points \( x \in I_1 \) and \( y \in I_2 \) be connected by an edge if and only if \( (x, y) \in R \). Then \( R \) contains a matching if and only if \( G \) contains a 1-factor. Also, \( G \) contains a 1-factor if and only if each connected component of \( G \) contains a 1-factor. Since every point of \( G \) has degree 1 or 2, the connected components of \( G \) are paths and circuits. (As for the notions of graph theory used above, see [1].) The circuits, infinite paths and finite paths of even length contain 1-factors, and hence, in order to find a 1-factor in \( G \) it is enough to prove that no component of \( G \) is a finite path of odd length.

Suppose this is not true and let \( C = \{ x_1, x_2, \ldots, x_n \} \) be a component of \( G \), where \( (x_{i-1}, x_i) \in R \) for every \( i = 2, \ldots, n \), and \( n \) is odd. Then \( x_1 \) and \( x_n \) both belong to \( I_1 \) or \( I_2 \) and have degree 1. Since the only elements of degree 1 in \( G \) are 0 and 1, this implies that \( x_1 = 0, x_n = 1 \) or \( x_1 = 1, x_n = 0 \).
If \((x, y) \in R\) and \((x, y)\) belongs to the segment joining \(A_0\) and \(A_1\) then \(y = -x + 2 - u\). If \((x, y)\) belongs to the other three segments then \(y = \pm x \pm u\). This implies, by induction on \(k\), that \(x_k = \pm x_1 + 2a_k + b_ku\) \((k = 2, \ldots, n)\), where \(a_k\) and \(b_k\) are integers. In particular, \(x_n = \pm x_1 + 2a_n + b_nu\), and hence \(b_nu = x_n \pm x_1 - 2a_n = \pm 1 - 2a_n\). Thus \(b_nu\) is an odd integer which implies \(b_n \neq 0\). Therefore \(u\) is rational, which contradicts our assumption. Hence \(G\) does not contain components of odd length and, consequently, contains a matching.

Let \(\mu\) denote the normalized linear measure on \(R\), i.e. let \(\mu(H) = \lambda_1(H)/2\sqrt{2}\) for every measurable \(H \subset R\). We define two maps, \(f\) and \(g\), of \(R\) into itself as follows. We put \(f(A_0) = A_0\) and \(f(A_2) = A_2\). If \((x, y) \in R\) and \(x \neq 0,1\) then there is a unique \(z \neq y\) such that \((x, z) \in R\), and we define \(f(x, y) = (x, z)\). Also, we define \(g(A_1) = A_1\) and \(g(A_3) = A_3\). If \((x, y) \in R\) and \(y \neq 0,1\) then there is a \(z \neq x\) such that \((z, y) \in R\) and we put \(g(x, y) = (z, y)\). It is easy to check that \(f\) and \(g\) are measure-preserving homeomorphisms of \(R\) onto itself with \(f^{-1} = f\) and \(g^{-1} = g\).

We prove that \(g \circ f\) is ergodic on \(R\), that is, if \(H\) is measurable and \(g \circ f(H) = H\) then \(\mu(H) = 0\) or \(\mu(H) = 1\).

Let \(T\) denote the circle group with the Lebesgue measure, and let \(h: R \to T\) be a measure-preserving homeomorphism of \(R\) onto \(T\) such that \(h(A_0) = 0\), \(h(A_1) = u/2\), \(h(A_2) = 1/2\), and \(h(A_3) = (1 + u)/2\). Let \(k = h \circ g \circ f \circ h^{-1}\), then \(k\) is a measure-preserving homeomorphism of \(T\) onto itself. Therefore either \(k(t) = t + c\) or \(k(t) = -t + c\) \((t \in T)\) with a constant \(c \in T\). It is easy to check that \(k(0) = u\) and \(k(1 - (u/2)) = u/2\) (see Figure 1). Hence \(k(t) = t + u\) for every \(t \in T\). Since \(u\) is irrational, \(k\) is ergodic on \(T\) (see [3, p. 26]), and hence \(g \circ f = h^{-1} \circ k \circ h\) is ergodic on \(R\). (This argument is similar to that in [8, p. 10].)
Now suppose that \( H \) is a measurable matching in \( R \). Then \( H \cap f(H) \) is finite and \( H \cup f(H) = R \). This implies \( \mu(H) = 1/2 \). Also, \( H \cap \sigma(H) \) is finite and \( H \cup \sigma(H) = R \), and hence the symmetric difference of the sets \( H \) and \( \sigma \circ f(H) \) is finite. Since \( \sigma \circ f \) is ergodic, this gives \( \mu(H) = 0 \) or \( \mu(H) = 1 \). This contradiction completes the proof.

Remark. It was proved by R. Rado in [7] that if the sections of a set \( K \subset I^2 \) are finite then \( K \) contains a matching if and only if \( \text{card} \{K_x : x \in H\} \geq \text{card} H \) and \( \text{card} \{K_y : y \in H\} \geq \text{card} H \) hold for every finite set \( H \subset I \).

Our theorem shows that Rado’s condition does not ensure the existence of a Borel matching even if \( K \) is compact. The same is true for König’s condition. Indeed, let \( R \) be a rectangle as in the theorem above, and let \( K \) be the union of \( R \) and the points \((0,0)\) and \((1,1)\). Then \( K \) is compact, \( \text{card} K_x = \text{card} K_y = 2 \) for every \( x, y \in I \) and \( K \) does not contain a measurable matching.

References


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