GROTHENDIECK GROUPS OF ALGEBRAS WITH NILPOTENT ANNIHILATORS
MAURICE AUSLANDER AND IDUN REITEN

(Communicated by Donald S. Passman)

ABSTRACT. Let $R$ be a commutative noetherian ring and $i: R \rightarrow \Lambda$ an $R$-algebra such that $\Lambda$ is a finitely generated $R$-module. Then the annihilator of $\Lambda$ in $R$ is nilpotent if and only if the cokernel of the induced map of Grothendieck groups $i^*: K_0(\text{mod } \Lambda) \rightarrow K_0(\text{mod } R)$ is a torsion group.

Let $k$ be an algebraically closed field, and let $S = k[[X_1, \ldots, X_n]]$ be the formal power series ring in $n$ variables over $k$. Let $G$ be a finite subgroup of $\text{GL}(n, k)$. Hence $G$ acts as a group of $k$-automorphisms of $S$, and we denote by $R$ the fixed ring $R = S^G$. Denote by $K_0(\text{mod } R)$ the Grothendieck group of the category of finitely generated $R$-modules $mod R$ modulo exact sequences. In [1] we proved that $K_0(\text{mod } R)$ is finitely generated. Under the additional assumption that $G$ acts freely we proved that $K_0(\text{mod } R)$ is isomorphic to $Z[\mathbb{R}] \oplus H$, where $H$ is a finite group and $[\mathbb{R}]$ denotes the image of $R$ in $K_0(\text{mod } R)$. The motivation for this paper was to show that the assumption that $G$ acts freely is not necessary. The proof we give here is based on the proof given in the Bielefeld May 1985 conference on representation theory and singularity theory. A different proof in the case that $G$ is abelian has been given in [3]. Our desired result is an easy consequence of the following general result which is also of independent interest. For the rest of the paper $R$ denotes a commutative noetherian ring and $\Lambda$ an $R$-algebra via a fixed map $i: R \rightarrow \Lambda$ such that $\Lambda$ is a finitely generated $R$-module.

THEOREM. The annihilator of $\Lambda$, $\text{ann}_R \Lambda$, is a nilpotent ideal in $R$ if and only if $\text{Coker}(K_0(\text{mod } \Lambda) \rightarrow K_0(\text{mod } R))$ is a torsion group.

This result is a direct consequence of the following two propositions.

PROPOSITION 1. If $\text{ann}_R \Lambda$ is nilpotent, then

$$\text{Coker}(K_0(\text{mod } \Lambda) \rightarrow K_0(\text{mod } R))$$

is torsion.

PROOF. Now, $\text{ann}_R \Lambda$ is nilpotent if and only if $\Lambda_p \neq 0$ for all prime ideals $p$ in $R$. From this characterization it follows easily that if $\mathfrak{A}$ is an ideal in $R$ and we consider the $R/\mathfrak{A}$-algebra $R/\mathfrak{A} \rightarrow \Lambda/\mathfrak{A} \Lambda$, then $\text{ann}_{R/\mathfrak{A}} \Lambda/\mathfrak{A} \Lambda$ is nilpotent. Assume that the proposition is false. Then because $R$ is noetherian there is an ideal $\mathfrak{A}$ in $R$ such that $\text{Coker}(K_0(\text{mod } \Lambda/\mathfrak{A} \Lambda) \rightarrow K_0(\text{mod } R/\mathfrak{A}))$ is not torsion, and
Coker\( (K_0(\text{mod } \Lambda/b\Lambda) \to K_0(\text{mod } R/b)) \) is torsion for any ideal \( \mathfrak{B} \) in \( R \) properly containing \( \mathfrak{A} \). We can clearly assume that \( \mathfrak{A} \) is zero.

Consider for a minimal prime ideal \( p \) in \( R \) the commutative exact diagram \([2, p. 642]\)

\[
\begin{array}{ccc}
\lim_{\mathfrak{p} \subseteq \mathfrak{p}} K_0(\text{mod } \Lambda/t\Lambda) & \longrightarrow & K_0(\text{mod } \Lambda) \\
\downarrow \alpha & & \downarrow i \\
\lim_{\mathfrak{p} \subseteq \mathfrak{p}} K_0(\text{mod } R/tR) & \longrightarrow & K_0(\text{mod } R) \\
\end{array}
\]

The snake lemma gives the exact sequence\( \text{Coker}\alpha \rightarrow \text{Coker}\beta \rightarrow \text{Coker}\gamma \rightarrow 0 \). Since \( R_p \) is a local artin ring, \( K_0(\text{mod } R_p) \simeq \mathbb{Z} \), and since \( \Lambda_p \neq 0 \), \( \beta \) is not zero, so that \( \text{Coker}\beta \) is torsion. By our assumption \( \text{Coker}(K_0(\text{mod } \Lambda/t\Lambda) \to K_0(\text{mod } R/tR)) \) is torsion since \( t \neq 0 \), and hence \( \text{Coker}\alpha \) is torsion. From this it follows that \( \text{Coker}\gamma \) is torsion, which contradicts our assumption, and the proof is done.

**PROPOSITION 2.** We have \( \text{rank}(\text{Coker } K_0(\text{mod } \Lambda \to K_0(\text{mod } R)) \geq n \), where \( n \) denotes the number of minimal primes in \( R \) not containing \( \mathfrak{A} = \text{ann}_R \Lambda \).

**PROOF.** Let \( p_1, \ldots, p_n, \ldots, p_r \) be the minimal primes in \( R \), where \( \mathfrak{A} \subseteq p_i \) for \( 1 \leq i \leq n \) and \( \mathfrak{A} \subset p_i \) for \( n < i \leq r \). Let \( T = R / \bigcup_{i=1}^{r} p_i \), and consider the diagram

\[
\begin{array}{ccc}
K_0(\text{mod } R/\mathfrak{A}) & \longrightarrow & K_0(\text{mod } (R/\mathfrak{A})_T) \\
\downarrow \delta & & \downarrow \gamma \\
K_0(\text{mod } R) & \longrightarrow & K_0(\text{mod } R_T) \\
\end{array}
\]

Since \( R_T \) is artin, \( K_0(\text{mod } R_T) \) is a free group with basis \( \{(R/p_i)_T; 1 \leq i \leq r\} \) and since \( (R/\mathfrak{A})_T \) is artin \( K_0(\text{mod } (R/\mathfrak{A})_T) \) is a free group with basis \( \{(R/p_i)_T; r \leq i \leq r\} \). Hence we have rank \( \text{Coker} \delta \geq \text{rank} \text{Coker} \gamma \geq n \). Since \( i: R \to \Lambda \) has the factorization \( R \to R/\mathfrak{A} \to \Lambda \), we get rank \( \text{Coker} \gamma \geq \text{rank} \text{Coker} \delta \geq n \).

We end this note with the following consequences of Proposition 1.

**COROLLARY 3.** If \( \Lambda \) is a commutative semilocal regular domain and \( R \subset \Lambda \), then \( \text{rank } K_0(\text{mod } R) = 1 \).

**PROOF.** Since \( \Lambda \) is regular, \( K_0(\text{mod } \Lambda) \simeq K_0(\mathcal{P}(\Lambda), 0) \), where \( \mathcal{P} \) denotes the category of finite generated projective \( \Lambda \)-modules and \( K_0(\mathcal{P}, 0) \) denotes the free group on the isomorphism classes of objects in \( \mathcal{P} \) modulo split exact sequences. Then \( K_0(\text{mod } \Lambda) \simeq \mathbb{Z} \) since every finitely generated projective \( \Lambda \)-module is free. Since \( R \subset \Lambda \), \( R \) is also a domain. If \( L \) denotes the quotient field of \( R \), we have a surjection \( K_0(\text{mod } R) \to K_0(\text{mod } L) \), so that rank \( K_0(\text{mod } R) \geq 1 \). It then follows from Proposition 1 that rank \( K_0(\text{mod } R) = 1 \).

**COROLLARY 4.** Let \( \Lambda \) be a commutative complete regular local domain and \( G \) a finite group acting on \( \Lambda \) as ring automorphisms, such that the order of \( G \) is invertible in \( \Lambda \). Then \( K_0(\text{mod } \Lambda^G) \simeq \mathbb{Z} \oplus H \), where \( H \) is a finite group.

**PROOF.** \( \Lambda \) is a finitely generated \( \Lambda^G \)-module \([4, \text{Corollary 5.9}] \) and \( R = \Lambda^G \) is noetherian \([4, \text{Corollary 1.12}] \). Then rank \( K_0(\text{mod } \Lambda^G) = 1 \) follows from Corollary 3, and it only remains to show that \( K_0(\text{mod } \Lambda^G) \) is finitely generated. This
follows as in [1, Proposition 3.4]: Since $\text{End}_{\Lambda G}(\Lambda) \simeq \Lambda^G$, there is a surjection $K_0(\text{mod } \Lambda G) \twoheadrightarrow K_0(\text{mod } \Lambda^G)$. Further $\Lambda G$ has finite global dimension since the order of $G$ is invertible in $\Lambda$, and hence $K_0(\text{mod } \Lambda G) \simeq K_0(\mathcal{P}(\Lambda G), 0)$. Since by our assumption on $\Lambda$ the Krull-Schmidt property holds for $\Lambda G$, $K_0(\mathcal{P}(\Lambda G), 0)$ is finitely generated.

REFERENCES

3. J. Herzog and H. Sanders, The Grothendieck group of invariant rings and of simple hypersurface singularities,