BANACH SPACE PROPERTIES
OF CIESIELSKI-POL’S C(K) SPACE

G. GODEFROY, J. PELANT, J. H. M. WHITFIELD AND V. ZIZLER
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ABSTRACT. A C(K) space X₀ which Ciesielski and Pol show does not continuously linearly inject into any c₀(Γ) has an equivalent C°°-norm, is Lipschitz equivalent to a c₀(Γ), and the density character of X₀ is equal to the w*-density character of X₀°.

1. Introduction. In nonseparable Banach space theory one of the important questions is that of injectability of the space into c₀(Γ). For example any weakly compactly generated Banach space or more generally any space analytic in its weak topology continuously linearly injects into c₀(Γ) [2, 12, 8]. In [9], Johnson and Lindenstrauss found the first example of a Banach space X with Fréchet differentiable norm that is not a subspace of any weakly compactly generated space. Their space continuously linearly injects into c₀ and the w*-density character of X* is N₀. Later on, Aharoni and Lindenstrauss proved in [1] that the space X of Johnson and Lindenstrauss is Lipschitz equivalent to a c₀(Γ) and thus provides a long sought example of two Lipschitz equivalent spaces that are not isomorphic. The space X of Johnson and Lindenstrauss contains a subspace Y which is isometrically isomorphic to c₀ and such that X/Y is isomorphic to c₀(Γ). We show that spaces with this property admit equivalent C°°-norms (Lemma 1).

Recently, Ciesielski and Pol have found an example X₀ of a space similar in structure to that of Johnson-Lindenstrauss space, which moreover has some additional striking properties [4]. Their space X₀ is a C(K) space where K is a “ladder system compact” (see definition below). Ciesielski and Pol show that X₀ does not linearly continuously inject into any c₀(Γ), while we show in this note that X₀ factors through its subspace which is isomorphic to c₀(Γ) to a space isometrically isomorphic to c₀(Γ₂). Following Aharoni and Lindenstrauss in [1] we then show that X₀ is Lipschitz equivalent to a c₀(Γ). X₀ has nice renorming properties: It admits an equivalent C∞-norm (Lemma 1), an equivalent locally uniformly rotund norm (since renormalizability by a locally uniformly rotund norm has the three space property). Since K(3), the third derived set of a compact K in the definition of X₀, is empty, the results of Deville [6] imply that X₀ has an equivalent norm ||·|| such that both ||·|| and its dual norm ||·||° on X₀° are locally uniformly rotund. Therefore the space X₀ allows us to sharpen a result of Dashiell and Lindenstrauss in [5], where...
the first example of strictly convex space which does not linearly continuously inject
into any $c_0(\Gamma)$ was found. The space of Dashiell and Lindenstrauss cannot have
an equivalent locally uniformly rotund norm since it contains an isomorph of $l_\infty$
[10, 11]. The space $X_0$ of Ciesielski and Pol not only does not linearly continu-
ously inject into any $c_0(\Gamma)$, but has a much stronger property (see the statement of
Theorem 1).

The notation in this note is standard in the Banach space theory. The spaces
are assumed to be real and differentiability is meant in a continuous Fréchet sense.
A Banach space $X$ is said to be Lipschitz equivalent to a Banach space $Y$ if there is
a (nonlinear) one-to-one map $T$ of $X$ onto $Y$ such that both $T$ and $T^{-1}$ satisfy the
Lipschitz condition. $\text{dens}X$ is the smallest cardinality such that there is a dense
set of $X$ of cardinality $\aleph$. $w^\ast$-dens $X^\ast$ is defined similarly. The set of all natural
numbers is denoted by $\omega$ and the cardinality of continuum by $c$.

2. Definition of Ciesielski-Pol's space and the main result. The following
definition is taken from [4, p. 686].

Let $B$ be a Bernstein set in the real line $R$ (i.e. for every perfect set $P \subset R$,
$P \cap B \neq \emptyset$ and $P \cap (R \setminus B) \neq \emptyset$).

Let $\{A_\alpha, \alpha < 2^\omega\}$ be an enumeration of all countable subsets of $R \setminus B$ with
uncountable closure. Choose by transfinite induction distinct points $a_\alpha \in A_\alpha \cap B$ and
for each $\alpha < 2^\omega$, choose and fix a sequence $C_\alpha = \{a_\alpha(i), i \in \omega\} \subset A_\alpha$
converging to $a_\alpha$. Let us give the real line $R$ the following locally compact topology:

Each point of $R \setminus \{a_\alpha\}_\alpha$ is isolated, while a base of neighborhoods of a point $a_\alpha$
are the sets $(a_\alpha \cup C_\alpha) \setminus F$, where $F$ are finite sets in $C_\alpha$. Finally, let $K = R \cup \infty$ be
the one-point compactification of $R$ with the given topology. Then $K$ is called a
ladder system compact and the space $C(K)$ of all real valued continuous functions
on $K$ endowed with the usual sup-norm will be called the Ciesielski-Pol space $X_0$.

The following theorem collects known information on $X_0$ together with the re-
results in this note.

**Theorem 1.** The space $X_0$ of Ciesielski and Pol defined above admits an equiv-
alent $C^\infty$-norm, is Lipschitz equivalent to a $c_0(\Gamma)$ and $\text{dens}X_0 = w^\ast$-dens $X^\ast = c$.

$X_0$ also has an equivalent norm $\| \cdot \|$ such that both $\| \cdot \|$ and its dual norm $\| \cdot \|^\ast$ on
$X_0^\ast$ are locally uniformly convex [6]. There is a subspace $Y \subset X_0$, $Y$ isomorphic to
a $c_0(\Gamma_1)$ such that $X_0/Y$ is isomorphic to a $c_0(\Gamma_2)$. However, from [4], if $(B_1,w)$
is the unit ball of $X_0$ endowed with the weak topology, then there is no continuous
one-to-one map of $(B_1,w)$ into any $c_0(\Gamma)$ endowed with its weak topology.

Let us recall that a norm $\| \cdot \|$ on a Banach space $X$ is called locally uniformly
rotund if $\lim \| x_n - x \| = 0$ whenever $x_n, x \in X$ are such that $\lim 2\| x \|^2 + 2\| x_n \|^2 -
\| x + x_n \|^2 = 0$.

3. Proofs of the results.

**Lemma 1.** Let $X$ be a Banach space and $Y$ be a subspace of $X$ such that $Y$ is
isomorphic to a space $c_0(\Gamma)$. Assume that $X/Y$ has an equivalent $C^k$-norm, where
$k \in \omega$ or $k = \infty$. Then $X$ admits an equivalent $C^k$-norm.

**Proof.** We use a technique of Kuiper whose construction of an equivalent
$C^\infty$-norm on $c_0(\Gamma)$ is shown in [3, p. 896].
First, by a proper extension of a norm given on $Y$ by its isomorphism to $c_0(\Gamma)$ we may suppose that $Y$ is isometric to $c_0(\Gamma)$. Let $T$ be a lifting operator of $Y^* \simeq X^*/Y^\perp \simeq l_1(\Gamma)$ into $X^*$, $\|T\| \leq 1$ and $\varphi_\alpha = T\delta_\alpha$, $\alpha \in \Gamma$, where $\delta_\alpha$ are the unit vectors of $l_1(\Gamma)$.

Let $\varphi$ be a real-valued $C^\infty$-function on the real line $R$ which is even, $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ for $|t| \leq 9/8$, $\varphi'(t) < 0$ for $t \in (9/8, 2)$, $\varphi(t) = 0$ for $|t| \geq 2$ and $\varphi$ is concave on the set $\{t \in R; \varphi(t) \geq 1/2\}$.

Let $\alpha(t)$ be a real-valued $C^\infty$-function on the real line $R$ such that $\alpha$ is even, $\alpha(t) = 0$ for $|t| \leq 1/4$, $\alpha(t) = 1$ for $|t| \geq 1/2$, $\alpha'(t) > 0$ for $t \in (1/4, 1/2)$ and $\alpha$ is convex on $\{t \in R; \alpha(t) \leq 1/2\}$. Let $\psi(\hat{x})$ be a function on $X/Y$ defined by $\psi(\hat{x}) = 1 - \alpha(\|\hat{x}\|)$, where $\hat{x}$ denotes the coset of $X/Y$ given by $x \in X$ and $\|\hat{x}\|$ is an equivalent $C^k$-norm on $X/Y$, $\|\cdot\|$ $\geq$ the original quotient norm of $X/Y$. Then the function $\psi$ on $X/Y$ is a $C^k$-function on $X/Y$. Let $\varphi(\hat{x}) = 1$ for $\|\hat{x}\| \leq 1/4$, $\varphi'(\hat{x}) = 0$ for $\|\hat{x}\| \geq 1/2$, $\varphi''(\hat{x})(\hat{x}) < 0$ for $1/4 < \|\hat{x}\| < 1/2$ and $\varphi$ is concave on the set $\{\hat{x} \in X/Y; \varphi(\hat{x}) \leq 1/2\}$. (Notice that $\varphi''(\hat{x})(\hat{x})$ denotes the Fréchet derivative of $\varphi$ at a point $\hat{x}$ in the direction $\hat{x}$.) Define a real-valued function $\Phi$ on $X$ by

$$\Phi(x) = \psi(\hat{x}) \prod_{\alpha \in \Gamma} \varphi(\varphi_\alpha(x)).$$

Finally, let $||| \cdot |||$ be the Minkowski functional of the set

$$S = \{x \in X; \Phi(x) \geq 1/2\}.$$
Clearly, \( S \) is a symmetric set in \( X \). We will now show that \( S \) is a convex set. To see this suppose that for \( x, y \in X, \Phi(x) \geq 1/2, \Phi(y) \geq 1/2 \) and \( t \in [0, 1] \). We will show that \( \Phi(tx + (1-t)y) \geq 1/2 \). First of all it is an elementary fact that if \( x_i \) and \( y_i \) are positive real numbers and \( t \in [0, 1] \), then if \( \prod x_i \geq 1/2 \) and \( \prod y_i \geq 1/2 \), then \( \prod (tx_i + (1-t)y_i) \geq 1/2 \). Therefore if \( x, y \in S \) and \( t \in [0, 1] \), then

\[
(*) \quad (t \psi(\hat{x}) + (1-t)\psi(\hat{y})) \prod_{\alpha \in \Gamma} \varphi(\varphi_\alpha(x)) + (1-t)\varphi(\varphi_\alpha(y)) \geq \frac{1}{2}.
\]

Since \( 0 \leq \varphi \leq 1 \) and \( 0 \leq \psi \leq 1 \), \( \Phi(x) \geq 1/2 \) and \( \Phi(y) \geq 1/2 \) imply that

\[
\psi(\hat{x}) \geq 1/2, \quad \psi(\hat{y}) \geq 1/2, \quad \varphi(\varphi_\alpha(x)) \geq 1/2 \quad \text{and} \quad \varphi(\varphi_\alpha(y)) \geq 1/2.
\]

Because of concavity of \( \varphi \) and \( \psi \) on the desired regions, it follows from (*) that

\[
\psi(t\hat{x} + (1-t)\hat{y}) \prod_{\alpha \in \Gamma} \varphi(\varphi_\alpha(tx + (1-t)y))
\]

\[
= \psi(t\hat{x} + (1-t)\hat{y}) \prod_{\alpha \in \Gamma} \varphi(t\varphi_\alpha(x) + (1-t)\varphi_\alpha(y))
\]

\[
\geq (t\psi(\hat{x}) + (1-t)\psi(\hat{y})) \prod_{\alpha \in \Gamma} t\varphi(\varphi_\alpha(x) + (1-t)\varphi(\varphi_\alpha(y)))
\]

\[
\geq 1/2.
\]

Therefore \( S \) is a convex set.

Finally, since \( |||x||| = \lambda > 0 \) such that \( \Phi(\lambda x) = 1/2 \), the Implicit Function Theorem guarantees that \( ||| \cdot ||| \) is a \( C^k \)-norm (away from the origin).

We omit the lengthy but straightforward check of this fact but notice that the denominator in the formula involved is not zero (at least one term in the product has its derivative by \( \lambda \) negative).

**Lemma 2.** If for a compact \( K, K^{(\omega)} \), the \( \omega \)th derived set of \( K \), is empty, then \( C(K) \) has an equivalent \( C^\infty \)-norm.

**Proof** It is enough to show, by induction on \( n \in \omega \), the following statement \( V_n \):

For every compact \( K \) such that \( K^{(n)} = \emptyset, C(K) \) has an equivalent \( C^\infty \)-norm.

If \( K^{(1)} = \emptyset \), then \( K \) is finite and \( C(K) \) has an equivalent \( C^\infty \)-norm.

Suppose now \( V_n \) is true and assume that \( K \) is a compact such that \( K^{(n+1)} = \emptyset \).

Consider the set \( E = \{ f \in C(K); f|K^{(1)} = 0 \}. E \) is a subspace of \( c_0(K) \) and as such has an equivalent \( C^\infty \)-norm (Kuiper [3, p. 896 or Lemma 1]).

Let \( Q \) be the restriction map of \( C(K) \) to \( C(K^{(1)}) \). The Tietze extension theorem guarantees that \( Q \) is a quotient map with the kernel \( E \). Thus \( C(K)/E \) is isomorphic to \( C(K^{(1)}) \). Since \( (K^{(1)})^{(n)} = \emptyset \), it follows from the induction hypothesis that \( C(K^{(1)}) \) has an equivalent \( C^\infty \)-norm. Lemma 1 can now be used to see that \( C(K) \) has an equivalent \( C^\infty \)-norm.

Following Aharoni and Lindenstrauss in [1] we get

**Lemma 3.** Let \( X_1 \) be the subspace of the Ciesielski-Pol space \( X_0 \) consisting of all \( f \in X_0 \) such that \( f(\infty) = 0 \). Then \( X_1 \) is Lipschitz equivalent to a \( c_0(\Gamma) \).

**Proof** (An adaptation of that in [1]). Let \( K \) be the ladder system compact in the definition of \( X_0 \). Then \( K^{(1)} = \{a_\alpha\}_\alpha \cup \infty \).
Let $E = \{f \in X_1; f|K^{(1)} \equiv 0\}$. Let $Q$ denote the restriction map of $X_1$ to $K^{(1)}$. Then Tietze extension theorem guarantees that $Q$ maps $X_1$ onto a subspace $X_2$ of $c_0(K^{(1)})$ formed by functions which are 0 at $\infty$ and that $X_1/E$ is isomorphic to $X_2$. $X_1/E$ is isomorphic to $X_2$.

We now use the idea of Aharoni and Lindenstrauss in [1] to find a lifting $\psi$ of $Q$, from $X_2$ to $X_1$. Let $y$ be a function from $X_2$ and

$$y = \sum_{n=1}^{t} a_n e_{\alpha_n} - \sum_{m=1}^{s} b_m e_{\alpha'_m}$$

with $a_1 \geq a_2 \geq a_3 \geq \cdots$, $b_1 \geq b_2 \geq b_3 \geq \cdots$, $0 \leq s, t \leq \infty$ be the unique representation of $y$ as a difference of disjointly supported positive elements $y^+$ and $y^-$ (assume that $\alpha_i \neq \alpha_j$ and $\alpha'_i \neq \alpha'_j$ for $i \neq j$) where $e_{\alpha_n}, e_{\alpha'_m}$ are the unit vectors in $c_0(K^{(1)}), \alpha_n, \alpha'_m \neq \infty$. Put

$$M_n = (\alpha_n \cup C_{\alpha_n})\setminus \left( \bigcup_{i=1}^{n-1} C_{\alpha_i}\right),$$

$$M'_m = (\alpha'_m \cup C_{\alpha'_m})\setminus \left( \bigcup_{j=1}^{m-1} C_{\alpha'_j}\right),$$

where $C_{\alpha_n}$ are the ladders for $\alpha_n$ in the definition of Ciesielski-Pol’s space $X_0$.

Furthermore, put for $y \in X_2$

$$\psi(y) = \sum_{n=1}^{t} a_n \chi_{M_n} - \sum_{m=1}^{s} b_m \chi_{M'_m},$$

where $\chi_{M_n}$ $(\chi_{M'_m})$ denotes the characteristic function of $M_n$ $(M'_m)$ in $K$. $\psi(y)$ is an extension of $y$ on $K$.

Now use the fact from [1] that if $y \in X_2$ and $k \in K\setminus\{a_\alpha\} \alpha$, then

$$\psi(y)(k) = \text{dist}(y^+, Z_k) - \text{dist}(y^-, Z_k),$$

where $Z_k = \text{sp}\{e_\alpha, k \notin C_\alpha\}$.

Therefore $\psi$ is a Lipschitz lifting of $Q$ and thus the Bartle-Graves map $Tu = (u - \psi(Qu), Qu)$ can be used to see that $X_1$ is Lipschitz equivalent to $X_1 \oplus E$.

Finally observe that $E$ is isometric to $c_0(K\setminus K^{(1)})$ (use the restriction map) and $X_2$ is isometric to $c_0(\{a_\alpha\} \alpha)$. Therefore $X_1$ is Lipschitz equivalent to $c_0(K\setminus K^{(1)}) \oplus c_0(\{a_\alpha\})$ and thus to a $c_0(\Gamma)$.

**Proof of Theorem 1.** First of all observe that the space $X_1$ from Lemma 3 is isomorphic to the Ciesielski-Pol space $X_0$. To see this fact consider a fixed element $a_\alpha$ in the definition of $X_0$ above and let $a_\alpha(1)$ be the first element of the ladder $C_\alpha$ corresponding to $a_\alpha$. The hyperplane $Z = \{f \in X_0; f(a_\alpha(1)) = 0\}$ is isomorphic to the space $X_0$, since it is clearly isomorphic to the space $C(K\setminus\{a_\alpha(1)\})$ (use the restriction map) and the space $C(K\setminus\{a_\alpha(1)\})$ is in turn isomorphic to $C(K)$ (use the shift on the ladder $C_\alpha$ and identity outside $C_\alpha$ to construct a homeomorphism of $K$ and $K\setminus\{a_\alpha(1)\}$). Since any two hyperplanes $Z_1, Z_2$ (through the origin) of a given Banach space $X$ are always isomorphic (both of them are isomorphic to $(Z_1 \cap Z_2) \oplus R$), $X_1$ is isomorphic to $X_0$. 

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Since for the ladder system compact $K$ in the definition of Ciesielski-Pol’s space $X_0$, $K^{(3)} = \emptyset$, the statement of renormability of $X_0$ by $C^\infty$-norm directly follows from Lemma 1.

The statement on factorization of $X_0$ is verified at the end of the proof of Lemma 3.

To show the statement about density characters, observe first that there are $c$ disjoint perfect sets in $R$ and since each of them intersects $K \setminus K^{(1)}$, it follows that the cardinality of $K \setminus K^{(1)}$ is equal to $c$. Furthermore, since $X_1$ is Lipschitz equivalent to $c_0(K \setminus K^{(1)}) \oplus c_0(\{a_0\})$ it follows that $\text{dens} X_0 = \text{dens} X_1 = c$. On the other hand, if $E = \{f \in C(K); f|K^{(1)} \equiv 0\}$, then

$$w^*-\text{dens} X_0^* = w^*-\text{dens} X_1^* \geq w^*-\text{dens} E^* = w^*(c_0(K \setminus K^{(1)})) = c$$

(cf. [10, Proposition 2.2]).

The statement that there is no continuous injection of $(B_1, w)$ into any $(c_0(\Gamma), \omega)$ was proved in [4, Proof of Lemma 5.2].

REMARKS AND OPEN PROBLEMS. 1. Assuming Generalized Continuum Hypothesis, Ciesielski and Pol show in [4] that a similarly constructed space $X_0' = C(K)$ where $K$ is a ladder system compact of ordinals, is Lindelöf in its weak topology yet having no weakly continuous injection into any $c_0(\Gamma)$ in its weak topology. One easily checks that most statements of this note remain valid for $X_0'$.

2. Due to the nonexistence of a weak-weak continuous injection of $X_1$ into any $c_0(\Gamma)$, the quotient space $X_2$ of $X_1$ has no weak-weak continuous lifting for a quotient map of $X_1$ onto $X_2$.

3. It would be interesting to decide if $X_0$ has an equivalent uniformly Gâteaux differentiable norm and if $X_0$ admits $C^\infty$-partitions of unity.

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REFERENCES


**Equipe d'Analyse, Université Paris VI, Place Jussieu, F-75230 Paris, France**

**Department of Mathematics, University of Missouri, Columbia, Missouri 65211**

**Mathematical Institute, Czechoslovak Academy of Sciences, Zitna 25, Prague, Czechoslovakia**

**Department of Mathematical Sciences, Lakehead University, Thunder Bay, Ontario, Canada P7B 5E1**

**Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1**