ABSTRACT. Though summability of a series by the Cesàro method $C_p$ does not in general imply its summability by the Borel-type method $(B, \alpha, \beta)$, it is shown that the implication holds under an additional condition.

1. Introduction. Suppose throughout that $\sum_{n=0}^{\infty} a_n$ is a series with partial sums $s_n := \sum_{k=0}^{n} a_k$, and that $\alpha > 0$ and $\alpha N + \beta > 0$ where $N$ is a nonnegative integer. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $(B, \alpha, \beta)$ to $s$ if

$$\alpha e^{-x} \sum_{n=N}^{\infty} \frac{x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} \to s \quad \text{as } x \to \infty.$$ 

The Borel-type summability method $(B, \alpha, \beta)$ is regular, and $(B, 1, 1)$ with $N = 0$ is the standard Borel summability method $B$.

We shall also be concerned with the Cesàro summability method $C_p$ ($p > -1$) and the Valiron method $V_\alpha$ defined as follows:

$$\sum_{n=0}^{\infty} a_n = s(C_p) \quad \text{if } c_n^p := \frac{s_n}{n^{p}} \to s \quad \text{as } n \to \infty$$

where

$$s_n := \sum_{k=0}^{n} \left( n - k + p - 1 \right) s_k;$$

$$\sum_{n=0}^{\infty} a_n = s(V_\alpha) \quad \text{if } \left( \frac{\alpha}{2\pi n} \right)^{1/2} \sum_{k=0}^{\infty} \exp \left( -\frac{\alpha(n-k)^2}{2n} \right) s_k \to s \quad \text{as } n \to \infty.$$ 

Consider the series $\sum_{n=1}^{\infty} a_n := \sum_{n=1}^{\infty} n^{a-1} \exp(A n a)$ where $A > 0$ and $0 < a < 1/2$. It is known [5, p. 213] that this series is summable $C_p$ for every $p > 0$ but is not convergent. However, since $a_n = o(n^{-1/2})$, it follows by the Borwein Tauberian Theorem [1, Theorem 1] that the series is not summable $(B, \alpha, \beta)$ for any $\alpha$ and $\beta$. This example shows that, in general, summability $C_p$ does not imply summability $(B, \alpha, \beta)$. The following theorem indicates how to strengthen the $C_p$ summability hypothesis in order to ensure summability $(B, \alpha, \beta)$.

**Theorem 1.** Suppose that $p$ is a nonnegative integer and that $c_n^p = s + o(n^{-p/2})$ as $n \to \infty$. Then $\sum_{n=0}^{\infty} a_n = s(B, \alpha, \beta)$.

The special case $\alpha = \beta = 1$, $p = 1$ of Theorem 1 has been proved by Hardy [5, Theorem 149]. Hardy and Littlewood [4, §3] proved that the condition
$c_n^p = s + o(n^{-1/2})$ is not sufficient for the summability of $\sum a_n$ by the Borel method. Hyslop [7, Theorem VIII] has obtained a more general result than Hardy, namely the case $\alpha = \beta = 1$ of Theorem 1. More recently, Swaminathan [10] has proved Theorem 1 with $p = 1$ and $(B, \alpha, \beta)$ summability replaced by the more general $F(a, q)$ summability introduced by Meir [9].

2. Preliminary results.

**Lemma 1** [8, Lemma 7]. Let $m < x_0 < n - 1$ where $m, n$ are integers and let the nonnegative function $f(x)$ be increasing on $[m, x_0]$ and decreasing on $[x_0, n]$. Then

$$\sum_{k=m}^{n} f(k) \leq \int_{m}^{n} f(x) dx + f(x_0).$$

**Lemma 2** [2, Theorem 3]. Suppose that $s_n = O(n^r)$ where $r \geq 0$. Then $\sum_{n=0}^{\infty} a_n = s(B, \alpha, \beta)$ if and only if $\sum_{n=0}^{\infty} a_n = s(V_\alpha)$.

**Theorem 2** (cf. [6, Theorem 2]). Suppose that $p$ is a nonnegative integer and that $c_n^p = s + o(n^{-p/2})$ as $n \to \infty$. Then $\sum_{n=0}^{\infty} a_n = s(V_\alpha)$.

**Proof.** Suppose, as we may without loss of generality, that $s = 0$. Let $v_n(z) := \exp(-\alpha(n - z)^2/2n)$ and denote the $p$th difference of $v_n(k)$ by $\Delta^p v_n(k)$, so that

$$\Delta^p v_n(k) = \sum_{r=0}^{p} \binom{p}{r} (-1)^r v_n(k + r).$$

Applying Abel's partial summation formula $p < m$ times, we have that

$$\sum_{k=0}^{m} s_k v_n(k) = \sum_{k=0}^{m-p} s_k^p \Delta^p v_n(k) + \sum_{r=0}^{p-1} s_{m-r}^r \Delta^r v_n(m - r).$$

Letting $m \to \infty$ and applying the limitation theorem for Cesàro summability [5, Theorem 46], we see that

$$F(n) := \sum_{k=0}^{\infty} s_k v_n(k) = \sum_{k=0}^{\infty} s_k^p \Delta^p v_n(k).$$

In order to prove the theorem we must show that $F(n) = o(n^{1/2})$. Since, by the hypothesis, $s_k^p = o(k^{p/2})$ as $k \to \infty$ and $k^{p/2} \Delta^p v_n(k) = o(n^{1/2})$ as $n \to \infty$, it suffices to show that

$$G(n) := n^{-1/2} \sum_{k=0}^{\infty} k^{p/2} |\Delta^p v_n(k)|$$

is bounded.

It is familiar that $\Delta^p v_n(k) = (-1)^p v_n^{(p)}(k + c)$ for some $c \in [0, p]$. Hence there is a $\theta = \theta(n, k) \in [0, p]$ such that

$$|\Delta^p v_n(k)| \leq |v_n^{(p)}(k + \theta)|.$$
Since \( v_n^{(p)}(x) = v_n(x) \sum_{0 \leq r \leq p/2} b_r (n-x)^{p-2r} n^{r-p} \), where the \( b_r \)'s are constants, we get from (1) and (2) that

\[
G(n) = O \left( \sum_{0 \leq r \leq p/2} |b_r| n^{r-p-1/2} \sum_{k=0}^{\infty} k^{p/2} |n - k - \theta|^{p-2r} v_n(k + \theta) \right).
\]

Therefore to establish that \( G(n) \) is bounded it is enough to show that, for \( 0 < r < p/2 \) and \( 0 < \theta < p \),

\[
H(n) := \sum_{k=0}^{\infty} k^{p/2} |n - k - \theta|^{p-2r} v_n(k + \theta) = O(n^{p-r+1/2}).
\]

Write

\[
H(n) = \left\{ \sum_{k=0}^{n-p-1} + \sum_{k=n-p}^{n} + \sum_{k=n+1}^{\infty} \right\} k^{p/2} |n - k - \theta|^{p-2r} v_n(k + \theta)
= \sum_1 \sum_2 + \sum_3.
\]

Since \( |n - k - \theta| \leq 2p \) for \( 0 \leq \theta \leq p \) and \( n - p < k < n \), and \( 0 < v_n(k + \theta) \leq 1 \), it is immediate that

\[
\sum_2 = O(n^{p/2}).
\]

Next, setting \( f(x) := x^{p-2r} \exp(-\alpha x^2/2n) \) and applying Lemma 1, we have that

\[
\sum_1 \leq \sum_{k=0}^{n-p-1} k^{p/2} (n-k)^{p-2r} v_n(k + p) \leq \sum_{k=p}^{n-1} k^{p/2} (n-k+p)^{p-2r} v_n(k)
\leq M n^{p/2} \sum_{k=p}^{n-1} f(n-k) \leq M n^{p/2} \sum_{k=1}^{n} f(k)
\leq M n^{p/2} \int_{1}^{n} f(x) \, dx + MC n^{p/2} \left( \frac{(p-2r)n}{\alpha} \right)^{p/2-r},
\]

where \( M := (1+p)^{p-2r} \) and \( C := \exp(r-p/2) \). Letting \( u = \alpha x^2/2n \), we get that

\[
\sum_1 = O \left( n^{p-r+1/2} \int_{1}^{\infty} u^{(p-1)/2-r} e^{-u} \, du \right) + O(n^{p-r}) = O(n^{p-r+1/2}).
\]

Further, with \( M \) and \( f(x) \) as above and \( g(x) := x^{3p/2-2r} \exp(-\alpha x^2/2n) \), we see that

\[
\sum_3 \leq \sum_{k=n+1}^{\infty} k^{p/2} (k-n+p)^{p-2r} v_n(k)
\leq M \left( \sum_{k=n+1}^{2n} + \sum_{k=2n+1}^{\infty} \right) k^{p/2} (k-n)^{p-2r} v_n(k)
\leq M (2n)^{p/2} \sum_{k=1}^{n} f(k) + M 2^{p/2} \sum_{k=n+1}^{\infty} g(k) := \sum_{3,1} + \sum_{3,2}.
\]
As above $\sum_{3,1} = O(n^{p-r+1/2})$. And finally, as $n \to \infty$,
\[
\sum_{3,2} = O \left( \int_n^\infty g(x) \, dx \right) + o(1) \\
= O \left( n^{3p/4-r+1/2} \int_{an/2}^\infty u^{3p/4-r-1/2} e^{-u} \, du \right) + o(1) \\
= o(1).
\]

Thus,
\[
(6) \quad \sum_{3} = O(n^{p-r+1/2}) + o(1) \quad \text{as } n \to \infty.
\]

It now follows from (3)-(6) that $H(n) = O(n^{p-r+1/2})$. This completes the proof. $\square$

3. Proof of Theorem 1. The limitation theorem for Cesàro summability [5, Theorem 46] implies that $s_n = o(n^p)$. Therefore, by Theorem 2 and Lemma 2, we have that $\sum_{n=0}^{\infty} a_n = s(B, \alpha, \beta)$. $\square$

4. Related results. The methods of Euler $E_\delta$, Meyer-Konig $S_\delta$, and Taylor $T_\delta$ ($0 < \delta < 1$) are defined as follows:
\[
\sum_{n=0}^{\infty} a_n = s(E_\delta) \quad \text{if} \quad \sum_{k=0}^{n} \binom{n}{k} \delta^k (1-\delta)^{n-k} s_k \to s \quad \text{as } n \to \infty;
\]
\[
\sum_{n=0}^{\infty} a_n = s(S_\delta) \quad \text{if} \quad (1-\delta)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} \delta^k s_k \to s \quad \text{as } n \to \infty;
\]
\[
\sum_{n=0}^{\infty} a_n = s(T_\delta) \quad \text{if} \quad (1-\delta)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} \delta^k s_{n+k} \to s \quad \text{as } n \to \infty.
\]

These methods, as well as the Borel-type and Valiron methods, are contained in the $F(a, q)$ family of methods mentioned in the introduction. The following theorem generalizes Swaminathan’s result [10], via Theorem 2 and [3, Satz III], for the Euler, Meyer-Konig, and Taylor methods.

**Theorem 3.** Suppose that $p$ is a nonnegative integer and that $c_n^p = s + o(n^{-p/2})$ as $n \to \infty$. Then for $0 < \delta < 1$, the series $\sum_{n=0}^{\infty} a_n$ is summable to $s$ by the $E_\delta$, $S_\delta$, and $T_\delta$ methods.

**References**


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