A COMMUTATOR ESTIMATE
FOR PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. For the commutator $B \cdot A - \text{Op}(ba)$ of two pseudo-differential operators $A$ and $B$ an estimate on weighted Sobolev spaces is proved under minimal regularity assumptions on the symbols $a$ and $b$.

1. Let $a, b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ two symbols. We impose the following conditions on $a$ and $b$. Let $N$ be a natural number and suppose that for all multi-indices $\alpha$ such that $|\alpha| \leq N$ there holds

\[ |D^{\alpha}b(x, \xi)| \leq C(1 + |\xi|)^{-|\alpha|}, \]
\[ |D_x, D^\alpha b(x, \xi)| \leq C \Omega(|\xi|)(1 + |\xi|)^{-|\alpha|} \]

if $i = 1, \ldots, n$. Suppose further that for all $\alpha$ such that $|\alpha| \leq n$ there holds

\[ |D^{\alpha}_x a(x, \xi)| \leq C(1 + |\xi|)^{-|\alpha|}, \]
\[ |D_x, D^\alpha a(x, \xi)| \leq C(1 + |\xi|)^{-|\alpha|}, \]
\[ |D_x, D^\alpha (a(x + h, \xi) - a(x, \xi))| \leq C\omega(|h|, |\xi|)(1 + |\xi|)^{-|\alpha|}. \]

We suppose that for each $t > 0$ the function $\omega(\cdot, t)$ is increasing and concave and that the functions $\omega(t, \cdot)$ and $\Omega$ are almost increasing in the following sense. There exists a positive constant $C$ independent of $t$ such that

\[ \omega(t, r) \leq C\omega(t, s) \]

whenever $0.5 * r \leq s \leq 2 * t$ and similarly for $\Omega$.

2. Let $w$ be a positive, locally integrable function. We say that $w \in A_p$, i.e. $w$ satisfies Muckenhoupt's $A_p$-condition for some $1 < p < \infty$, iff

\[ \sup \frac{1}{|Q|} \int_Q w \left( \frac{1}{|Q|} \int_Q w^{-1/(p-1)} dx \right)^{p-1} < \infty \]

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Denote by $L^p(w)$ the weighted $L^p$-space and let $J^s$ be the Bessel potential of order $s \in \mathbb{R}$. The weighted Sobolev space $H^{s,p}(w)$ is defined to be the space of all tempered distributions $f$ such that

\[ \|f\|_{H^{s,p}(w)} := \|J^{-s}f\|_{L^p(w)} < \infty \]

(compare Miler [7]).

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3. Our main objective is to prove the following result.

**Theorem 1.** Let $N = n + \lfloor n/2 \rfloor + 2$ and suppose that the symbols $a$ and $b$ satisfy (1), (2) and (3), (4), (5) respectively. Let $\omega$ and $\Omega$ satisfy (6) and be such that $\{2^{-j}\Omega(2^j)\}$, $\{\omega(2^{-j}, 2^j)\} \in l^2(N)$. Suppose that $1 < p < \infty$, $w \in A_p$ and $0 < s < 1$. Then the commutator

$$B \cdot A - \text{Op}(ab): H^{s-1,p}(w) \to H^{s,p}(w)$$

is bounded. \( \square \)

This theorem extends earlier results by Kumano Go and Nagase [3] and Bourdaud [1]; see also Marschall [4].

4. Before we prove the theorem let us provide a result needed in the proof. Let $c: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a symbol and suppose that for all multi-indices $\alpha$ such that $|\alpha| \leq n$ there holds

$$|D_{\xi}^\alpha c(x, \xi)| \leq C(1 + |\xi|)^{-|\alpha|},$$

$$|D_{\xi}^\alpha (c(x + h, \xi) - c(x, \xi))| \leq C\tilde{\omega}(|h|, |\xi|)(1 + |\xi|)^{-|\alpha|}. $$

**Proposition 2.** Let $1 < p < \infty$ and $w \in A_p$.

(i) If the symbol $c$ satisfies (9) and (10) with a function $\tilde{\omega}$ such that (6) and $\{\tilde{\omega}(2^{-j}, 2^j)\} \in l^2(N)$ hold, then the operator $C: L^p(w) \to L^p(w)$ is bounded.

(ii) Suppose that (9) holds and that for each $\xi \in \mathbb{R}^n$ the function $c(\cdot, \xi)$ has its spectrum contained in the ball $\{\eta: |\eta| \leq 0.1 * (1 + |\xi|^2)^{1/2}\}$. Then for every real number $s$ the operator $C: H^{s,p}(w) \to H^{s,p}(w)$ is bounded. \( \square \)

For a proof of the proposition see Marschall [5] and also Coifman and Meyer [2].

5. We are now in the position for the **Proof of the Theorem.**

**Step (i).** Let $K$ be a function belonging to the Schwartz space $S(\mathbb{R}^n)$ such that the spectrum of $K$ is contained in the ball $B(0, 0.05)$ and that the Fourier transform of $K$ is equal to one in a neighborhood of the origin. Define

$$K_\xi(x) := (1 + |\xi|^2)^{n/2}K((1 + |\xi|^2)^{1/2}x)$$

and decompose the symbols $a$ and $b$ as follows. Let

$$a_1(x, \xi) := \int K_\xi(y)a(x - y, \xi) dy,$$

$$b_1(x, \xi) := \int K_\xi(y)b(x - y, \xi) dy$$

and $a_2 := a - a_1$ and $b_2 := b - b_1$. Note that by the conditions on the Fourier transform of $K$ one has

$$\int K_\xi(y) dy = 1, \quad \int y_i K_\xi(y) dy = 0, \quad i = 1, \ldots, n.$$

Hence, it follows that

$$|D_\xi^\alpha b_2(x, \xi)| \leq C * \Omega(|\xi|)(1 + |\xi|)^{-1-|\alpha|},$$

$$|D_x D_\xi^\alpha b_2(x, \xi)| \leq C * \Omega(|\xi|)(1 + |\xi|)^{-|\alpha|},$$

$$|D_\xi^\alpha a_2(x, \xi)| \leq C\omega((1 + |\xi|)^{-1}, |\xi|)(1 + |\xi|)^{-1-|\alpha|},$$

$$|D_x D_\xi^\alpha a_2(x, \xi)| \leq C\omega((1 + |\xi|)^{-1, |\xi|}(1 + |\xi|)^{-|\alpha|}. $$
For example, in order to prove (13) observe that
\[ a_2(x, \xi) = \int K_\xi(y) \left( a(x, \xi) - a(x - y, \xi) - \sum_{i=1}^{n} y_i \frac{\partial a}{\partial x_i}(x, \xi) \right) dy \]
and hence, by the mean value theorem
\[
|a_2(x, \xi)| \leq C \int |y| |K_\xi(y)|\omega(|y|, |\xi|) dy \\
\leq C(1 + |\xi|)^{-1} \int |y| |K(y)|\omega((1 + |\xi|)^{-1}|y|, |\xi|) dy.
\]
Now, \( \omega(\cdot, |\xi|) \) being increasing and concave, one has
\[
\omega((1 + |\xi|)^{-1}|y|, |\xi|) \leq (1 + |y|)\omega((1 + |\xi|)^{-1}, |\xi|)
\]
(compare Coifman and Meyer [2]) and (13) follows.

**Step (ii).** By complex interpolation it suffices to prove the theorem in the endpoint cases \( s = 0 \) and \( s = 1 \). Consider first the term \( C_i := B_i \cdot A_1 - \text{Op}(b_i a_1), \ i = 1, 2 \). It has the symbol
\[
C_i(x, \xi) = \frac{1}{(2\pi)^n} \sum_{j=1}^{n} \int_{0}^{1} \int \frac{\partial b_i}{\partial \xi_j}(x, \xi + t\eta)(D_{x_j} a_1)(\eta, \xi) d\eta dt
\]
where \( (D_{x_j} a_1)(\cdot, \xi) \) is meant to be the Fourier transform of the function \( D_{x_j} a_1(\cdot, \xi) \).

Then, using a method by Meyer [6], it follows that for all multi-indices \( \alpha \) such that \( |\alpha| \leq n \) one has
\[
|D_\xi^\alpha c_1(x, \xi)| \leq C(1 + |\xi|)^{-|\alpha|}, \\
|D_\xi^\alpha c_2(x, \xi)| \leq C \Omega(|\xi|)(1 + |\xi|)^{-2-|\alpha|}, \\
|D_{x_j} D_\xi^\alpha c_2(x, \xi)| \leq C \Omega(|\xi|)(1 + |\xi|)^{-1-|\alpha|}.
\]

But then the proposition and the condition \( \{2^{-j}\Omega(2^j)\} \in l^2(\mathbb{N}) \) yield the boundedness of
\[
C_i: H^{s-1,p}(w) \to H^{s,p}(w).
\]

**Step (iii).** Observe that part (i) of the proposition and the condition \( \{2^{-j}\Omega(2^j)\} \in l^2(\mathbb{N}) \) imply
\[
B: L^p(w) \to L^p(w), \\
B: H^{1,p}(w) \to H^{1,p}(w).
\]

Further, the proposition and (13), (14) yield
\[
A_2: H^{-1,p}(w) \to L^p(w), \\
A_2: L^p(w) \to H^{1,p}(w)
\]
and consequently we get the boundedness of the term \( B \cdot A_2 \). Since the remaining term \( \text{Op}(ba_2) \) is treated similarly, the theorem is proved completely. \( \square \)

6. Let us remark that one can introduce symbols \( b \) with \( N \) not an integer. Then using complex interpolation of symbols one can see that the theorem holds when \( N = 3n/2 + 1 \). For such techniques we refer to Marschall [4].
REFERENCES


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