A NOTE ON THE GENERALIZED RIEMANN INTEGRAL

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ABSTRACT. We show that the generalized Riemann integral can be defined by means of gage functions which are upper semicontinuous when restricted to a suitable subset whose complement has measure zero.

By introducing $\delta$-fine partitions for a positive function $\delta$ (see below), Henstock and Kurzweil obtained a strikingly simple Riemannian definition of the Denjoy-Peron integral (cf. Definition 1 and [S, Chapter VIII]). In their definition, the function $\delta$ is completely arbitrary, and it is not clear how complicated it need be (a question of P. S. Bullen—see [Q]). The purpose of this note is to establish that $\delta$ can be always selected so that it is upper semicontinuous when restricted to a suitable subset whose complement has measure zero (cf. [P2, Lemma 3]). The proof is quite simple: we show first that such a $\delta$ can be chosen if the integrand is Lebesgue integrable, and then we follow the constructive Denjoy definition, observing that the upper semicontinuity property of $\delta$ is preserved at the inductive step. We also show that for a bounded Lebesgue integrable function, a gage $\delta$ can be selected so that it is upper semicontinuous everywhere (cf. [FM, Example 1]).

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By $\mathbb{R}$ and $\mathbb{R}_+$ we denote the set of all real and all positive real numbers, respectively. Unless stated otherwise, all functions in this paper are real-valued. When no confusion is possible, we denote by the same symbol a function on a set $E$, as well as its restrictions to various subsets of $E$. An interval is a compact nondegenerate subinterval of $\mathbb{R}$. A collection of intervals whose interiors are disjoint is called a nonoverlapping collection. If $E \subset \mathbb{R}$, then $\text{cl}(E)$, $\text{int}(E)$, $d(E)$, and $|E|$ denote, respectively, the closure, interior, diameter, and outer Lebesgue measure of $E$. A function $\delta$ on an interval $A$ is called nearly upper semicontinuous if there is a set $H \subset A$ such that $|A - H| = 0$ and $\delta \upharpoonright H$ is upper semicontinuous.

A subpartition of an interval $A$ is a collection $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ where $A_1, \ldots, A_p$ are nonoverlapping subintervals of $A$, and $x_i \in A_i$, $i = 1, \ldots, p$. If, in addition, $\bigcup_{i=1}^{p} A_i = A$, we say that $P$ is a partition of $A$. Given a $\delta: A \to \mathbb{R}_+$, we say that a subpartition $P$ is $\delta$-fine whenever $d(A_i) < \delta(x_i)$ for $i = 1, \ldots, p$. An easy compactness argument shows that a $\delta$-fine partition of an interval $A$ exists for each $\delta: A \to \mathbb{R}_+$.
If \( f \) is a function on an interval \( A \) and \( P = \{(A_1, x_1), \ldots, (A_p, x_p)\} \) is a subpartition of \( A \), we let
\[
\sigma(f, P) = \sum_{i=1}^{p} f(x_i)|A_i|.
\]

1. DEFINITION (HENSTOCK-KURZWEIL). A function \( f \) on an interval \( A \) is called integrable in \( A \) if there is a real number \( I \) with the following property: given \( \varepsilon > 0 \), we can find a \( \delta : A \to \mathbb{R}_+ \) such that \( |\sigma(f, P) - I| < \varepsilon \) for each \( \delta \)-fine partition \( P \) of \( A \).

Since \( \delta \)-fine partitions of an interval \( A \) exist for each \( \delta : A \to \mathbb{R}_+ \), it is easy to see that the number \( I \) from the previous definition is determined uniquely by the integrable function \( f \). It is called the integral of \( f \) over \( A \), denoted by \( \int_A f \), or \( \int_a^b f \) if \( A = [a, b] \). The family of all integrable functions on \( A \) is denoted by \( \mathcal{R}(A) \).

A detailed study of the integral defined above can be found in [H and K]; an elementary exposition is given in [Ml and P1]. In particular, it is shown in [K, Theorem 4.14 and P1, Corollary B5] that the integral coincides with the Denjoy-Perron integral (see [S, Chapter VIII, Theorems (3.9) and (3.11)]). As this result is important for our purposes, we formulate it precisely.

The family of all Denjoy-Perron integrable functions on an interval \( A \) is denoted by \( \mathcal{D}(A) \), and if \( f \in \mathcal{D}(A) \), the symbol \( (D) \int_A f \) denotes the Denjoy-Perron integral of \( f \) over \( A \).

2. THEOREM (HENSTOCK-KURZWEIL). If \( A \) is an interval, then \( \mathcal{R}(A) = \mathcal{D}(A) \) and \( \int_A f = (D) \int_A f \) for each \( f \in \mathcal{R}(A) \).

The function \( \delta \) from Definition 1 is often referred to as a gage associated to \( f \) and \( \varepsilon \). For an integrable function \( f \) on an interval \( A \) and an \( \varepsilon > 0 \), we denote by \( \Delta(f, A; \varepsilon) \) the family of all gage functions associated to \( f \) and \( \varepsilon \). Since positive continuous functions on compact intervals are bounded away from zero, we see immediately that \( f \) is Riemann integrable in the classical sense if and only if \( \Delta(f, A; \varepsilon) \) contains a continuous gage for each \( \varepsilon > 0 \). Our goal is to show that for each \( \varepsilon > 0 \), the family \( \Delta(f, A; \varepsilon) \) always contains a nearly upper semicontinuous gage. To this end, we denote by \( \mathcal{R}^*(A) \) the family of all \( f \in \mathcal{R}(A) \) such that \( \Delta(f, A; \varepsilon) \) contains a nearly upper semicontinuous function for each \( \varepsilon > 0 \), and we show that \( \mathcal{R}^*(A) = \mathcal{R}(A) \).

3. LEMMA. Let \( h \) be a lower semicontinuous function on a set \( E \subset \mathbb{R} \), let \( \eta > 0 \), and for each \( x \in E \), let \( \delta(x) \) be the supremum of all numbers \( \delta \in (0, 1] \) such that \( y \in E \) and \( |y - x| < \delta \) implies \( h(y) \geq h(x) - \eta \). Then the function \( x \mapsto \delta(x) \) is upper semicontinuous on \( E \).

PROOF. Proceeding towards a contradiction, suppose that there is an \( x \in E \) and a sequence \( \{x_n\} \) in \( E \) such that \( x_n \to x \) and \( \lim \delta(x_n) > \delta(x) + \alpha \) for some \( \alpha > 0 \). By the definition of \( \delta(x) \), there is a \( y \in E \) with \( |y - x| < \delta(x) + \alpha/2 \) and \( h(y) < h(x) - \eta \). Choose a \( \beta > 0 \) so that \( h(y) < h(x) - \eta - \beta \), and find an \( x_n \) for which \( |x - x_n| < \alpha/2 \), \( \delta(x_n) > \delta(x) + \alpha \), and \( h(x_n) \geq h(x) - \beta \). Then
\[
|y - x_n| \leq |y - x| + |x - x_n| < \delta(x) + \alpha < \delta(x_n),
\]
and hence \( h(y) \geq h(x_n) - \eta \geq h(x) - \eta - \beta \), a contradiction.

If \( E \) is a Lebesgue measurable subset of \( \mathbb{R} \), we denote by \( \mathcal{L}^*(E) \) the family of all functions \( f \) on \( E \) for which the finite Lebesgue integral \( (L) \int_E f \) exists.
4. PROPOSITION. If $A$ is an interval, then $\mathcal{L}(A) \subset \mathcal{R}_+(A)$.

PROOF. Let $f \in \mathcal{L}(A)$, $\varepsilon > 0$, and let $\eta = \varepsilon/(|A| + 1)$. There is an upper semicontinuous function $g: A \to [-\infty, +\infty)$, and a lower semicontinuous function $h: A \to (-\infty, +\infty]$ such that $g \leq f \leq h$ and $(L) \int_A (h - g) < \eta$. Let $E = \{x \in A: h(x) < +\infty\}$, and let $\delta_h$ be the positive upper semicontinuous function on $E$ associated to $h$ and $\eta$ according to Lemma 3. If $x \in A - E$, we select any $\delta_h(x) > 0$ so that $h(y) \geq f(x) - \eta$ for each $y \in A$ with $|y - x| < \delta_h(x)$. Since $|A - E| = 0$, we have defined a nearly upper semicontinuous function $\delta_h: A \to \mathbb{R}_+$. Using $-g$ instead of $h$, we define similarly a nearly upper semicontinuous function $\delta_g: A \to \mathbb{R}_+$, and set $\delta = \min(\delta_h, \delta_g)$. Now if $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a $\delta$-fine partition of $A$, then $g(x) \leq f(x_i) + \eta$ and $h(x) \geq f(x_i) - \eta$ for each $x \in A_i$, $i = 1, \ldots, p$. Thus $(L) \int_{A_i} g \leq (L) \int_{A_i} f \leq (L) \int_{A_i} h$, and

\[
(L) \int_{A_i} g - \eta |A_i| \leq f(x_i) |A_i| \leq (L) \int_{A_i} h + \eta |A_i|, \quad i = 1, \ldots, p.
\]

It follows that

\[
\left| \sigma(f, P) - (L) \int_A f \right| \leq \sum_{i=1}^p \left| f(x_i) |A_i| - (L) \int_{A_i} f \right| \leq \sum_{i=1}^p \left[ \eta |A_i| + (L) \int_{A_i} (h - g) \right] < \varepsilon,
\]

and we have $f \in \mathcal{R}_+(A)$.

If $f \in \mathcal{L}(A)$ then, in general, $\Delta(f, A; \varepsilon)$ may contain no gage which is a Baire functions on the whole interval $A$ (see [FM, Example 1]). However, a closer look at the proof of Proposition 4 shows that $\Delta(f, A; \varepsilon)$ contains an upper semicontinuous gage for each $\varepsilon > 0$ whenever $f$ has an upper semicontinuous majorant and a lower semicontinuous minorant which are both finite. In particular, we have the following corollary.

5. COROLLARY. If $f$ is a bounded Lebesgue integrable function on an interval $A$, then $\Delta(f, A; \varepsilon)$ contains an upper semicontinuous gage for every $\varepsilon > 0$.

6. REMARK. The proofs of Proposition 4 and Corollary 5 translate verbatim to the higher dimensional Henstock-Kurzweil integrals, as well as to the integral defined by McShane in [Ms]. Since the McShane integral coincides with that of Lebesgue (see [P_1, Corollary B11 or M_1, §8.3]), we see that it can be always defined by means of nearly upper semicontinuous gages, which can be taken upper semicontinuous whenever the integrand is bounded. This remains true even for a general setting discussed in [AP], provided the underlying space is metrizable.

7. LEMMA. Let $A = [a, b]$ be an interval.

(i) The family $\mathcal{R}_+(A)$ is a real vector space.

(ii) If $f \in \mathcal{R}_+(A)$, then $f \in \mathcal{R}_+(B)$ for each subinterval $B$ of $A$.

(iii) If $f \in \mathcal{R}_+(A)$ and $\varepsilon > 0$, then there is a nearly upper semicontinuous function $\delta: A \to \mathbb{R}_+$ such that

\[
\sum_{i=1}^p \left| f(x_i) |A_i| - \int_{A_i} f \right| < \varepsilon
\]

for each $\delta$-fine subpartition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ of $A$. 

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(iv) If \( c \in (a, b) \) and \( f : A \rightarrow \mathbb{R} \) belongs to \( \mathcal{R}_*([a, c]) \) and \( \mathcal{R}_*([c, b]) \), then \( f \) belongs also to \( \mathcal{R}_*([a, b]) \).

(v) If \( f : A \rightarrow \mathbb{R} \) belongs to \( \mathcal{R}_*([a, c]) \) for each \( c \in (a, b) \), and a finite limit
\[
\lim_{c \to a^+} \int_c^b f = I
\]
exists, then \( f \in \mathcal{R}_*([a, b]) \) and \( \int_a^b f = I \).

**Proof.** The proofs of properties (i)–(v) are the same as those of the corresponding properties of the Henstock-Kurzweil integral (cf. [M], §§2.1, 2.3, S3.7, 2.4, and S2.8). We only need to observe two facts:

1. A function which is equal almost everywhere to a nearly upper semicontinuous function is itself nearly upper semicontinuous.
2. The distance function from a subset of \( \mathbb{R} \) is continuous.

For illustration, we sketch a fairly complicated proof of property (v), following the pattern of [P, Theorem A7].

Choose an \( \varepsilon > 0 \), and find a \( \gamma \in (a, b) \) so that \( |f_a f - I| < \varepsilon /3 \) for each \( c \in [\gamma, b) \), and \( |f(b)(b - \gamma)| < \varepsilon /3 \). Select a strictly increasing sequence \( \{c_n\}_{n=0}^\infty \) in \( [a, b) \) with \( c_0 = a \) and \( \lim c_n = b \). By (iii), for each \( n = 1, 2, \ldots \), there is a nearly upper semicontinuous \( \delta_n : [c_{n-1}, c_n] \to \mathbb{R}_+ \) such that
\[
\sum_{i=1}^p |f(x_i)|A_i - \int_{A_i} f < \frac{\varepsilon}{3} 2^{-n}
\]
whenever \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) is a \( \delta_n \)-fine subpartition of \( [c_{n-1}, c_n] \). In view of observations (1) and (2), we may assume that
\[
\delta_n(x) \leq \min(|x - c_{n-1}|, |x - c_n|)
\]
for each \( x \in (c_{n-1}, c_n) \), \( \delta_1(c_0) \leq c_1 - c_0 \), and \( \delta_n(c_n) = \delta_{n+1}(c_n) \leq \min(c_n - c_{n-1}, c_{n+1} - c_n) \), \( n = 1, 2, \ldots \).

Clearly, the function \( \delta \) on \( A \) defined by
\[
\delta(x) = \begin{cases} 
\delta_n(x) & \text{if } x \in [c_{n-1}, c_n], \ n = 1, 2, \ldots, \\
b - \gamma & \text{if } x = b,
\end{cases}
\]
is positive and nearly upper semicontinuous. We show that it belongs to \( \Delta(f, A; \varepsilon) \).

Let \( P = \{(A_{i, 1}, x_1), \ldots, (A_{i, p}, x_p)\} \) be a \( \delta \)-fine partition of \( A \). After a suitable reordering, we may assume that \( A_i = [t_{i-1}, t_i], \ i = 1, \ldots, p \), where \( a = t_0 < \ldots < t_p = b \). Replacing \( (A_i, x_i) \) in \( P \) by \( \{(t_{i-1}, x_i), (x_i, t_i), (x_i, x_{i+1})\} \), whenever \( x_i \in (t_{i-1}, t_i) \), we obtain a \( \delta \)-fine partition \( Q \) of \( A \) with \( \sigma(f, Q) = \sigma(f, P) \). Thus with no loss of generality we may also assume that \( x_i = t_{i-1} \) or \( x_i = t_i \) for each \( i = 1, \ldots, p \). From this and the choice of \( \delta \), we make the following conclusion: if
\[
P_n = \{(A_{i, 1}, x_1) \in P : A_i \subset [c_{n-1}, c_n]\}
\]
for \( n = 1, 2, \ldots \), and if \( N \) is the first positive integer with \( c_N \geq t_{p-1} \), then conditions (a)–(c) below are satisfied.

(a) \( P_n \) is a \( \delta_n \)-fine partition of \( [c_{n-1}, c_n] \) for \( n = 1, \ldots, N - 1 \).

(b) \( P_N \) is a \( \delta_N \)-fine partition of \( [c_{N-1}, t_{p-1}] \); in particular, \( P_N \) is a \( \delta_N \)-fine subpartition of \( [c_{N-1}, c_N] \).
Now we have
\[
|\sigma(f, P) - I| \leq \left| \int_a^{t_{p-1}} f \right| + \left| \int_{c_{n-1}}^{c_n} f \right| < \sum_{n=1}^{N} \left| \sigma(f, P_n) - \int_{c_n}^{c_{n-1}} f \right| + \left| \sigma(f, P_n) - \int_{c_{n-1}}^{c_{n-1}} f \right| + |f(b)|(b - t_{p-1}) + \frac{\varepsilon}{3} \sum_{n=1}^{N} 2^{-n} + 2 \cdot \frac{\varepsilon}{3} < \varepsilon,
\]
and the proof is completed.

8. LEMMA. Let \( f \) be a function on an interval \( A \), and let \( \{B_n : n = 1, 2, \ldots\} \) be a disjoint family of subintervals of \( A \) such that \( f \in \mathcal{R}_*(B_n) \) for \( n = 1, 2, \ldots \), and \( f \in \mathcal{L}(A - \bigcup_{n \geq 1} B_n) \). Further let \( W_n = \sup |f| \) where the supremum is taken over all intervals \( C \subset B_n \), and suppose that \( \sum_{n \geq 1} W_n < +\infty \). Then \( f \in \mathcal{R}_*(A) \).

PROOF. Let \( S = A - \bigcup_{n \geq 1} B_n \) and \( I = (L) \int_S f + \sum_{n \geq 1} \int_{B_n} f \). Since \( |\int_{B_n} f| \leq W_n \), we see that \( I \) is a well-defined real number. Choose an \( \varepsilon > 0 \), and find an integer \( N \geq 1 \) with \( \sum_{n > N} W_n < \varepsilon/6 \). Let \( G = \bigcup_{n > N} \text{int}(B_n) \), \( T = A - (G \cup \bigcup_{n=1}^{N} B_n) \), and let \( \varphi, \psi, \) and \( h \) be, respectively, the functions \( f \upharpoonright \bigcup_{n=1}^{N} B_n \), \( f \upharpoonright T \), and \( f \upharpoonright G \) extended to \( A \) by zero. Thus \( f = \varphi + \psi + h \), and it follows from Proposition 4 and Lemma 7 (iv) that \( \varphi \in \mathcal{R}_*(A) \) and \( \int_A \varphi = \sum_{n=1}^{N} \int_{B_n} f \). Since \( T \) differs from \( S \) only by a countable set, Proposition 4 implies that \( \psi \in \mathcal{R}_*(A) \), and we have \( \int_A \psi = (L) \int_S f \). Hence by Lemma 7, (i), the function \( g = \varphi + \psi \) belongs to \( \mathcal{R}_*(A) \). Consequently, we can find a nearly upper semicontinuous function \( \delta_g : A \to \mathbb{R}_+ \) so that
\[
|\sigma(g, P) - (L) \int_S f + \sum_{n=1}^{N} \int_{B_n} f| < \frac{\varepsilon}{3}
\]
for each \( \delta_g \)-fine partition \( P \) of \( A \). By Lemma 7, (iii), there is a nearly upper semicontinuous function \( \delta_n : B_n \to (0, 1] \) such that
\[
\sum_{i=1}^{q} |f(z_i)|E_i| - \int_{E_i} f| < \frac{\varepsilon}{3} 2^{-n}
\]
for each \( \delta_n \)-fine subpartition \( \{(E_1, z_1), \ldots, (E_q, z_q)\} \) of \( B_n \), \( n = 1, 2, \ldots \). In view of observation (2) in the proof of Lemma 7, we may assume that \( (x - \delta_n(x), x + \delta_n(x)) \subset B_n \) whenever \( x \in \text{int}(B_n) \). We define a nearly upper semicontinuous function \( \delta_n : A \to (0, 1] \) by setting
\[
\delta_n(x) = \begin{cases} 
\delta_n(x) & \text{if } x \in B_n \text{ and } n > N, \\
1 & \text{otherwise},
\end{cases}
\]
and we show that \( \delta = \min(\delta_g, \delta_n) \) belongs to \( \Delta(f, A; \varepsilon) \).

To this end, let \( P = \{(A_1, z_1), \ldots, (A_p, z_p)\} \) be a \( \delta \)-fine partition of \( A \). For \( n = 1, 2, \ldots \), denote by \( K_n \) the set of all the integers \( k \) with \( 1 \leq k \leq p \) and \( z_k \in \text{int}(B_n) \), and set \( P_n = \{(A_k, z_k) \in P : k \in K_n\} \) and \( C_n = \text{cl}(B_n - \bigcup_{k \in K_n} A_k) \). Then each \( C_n \) is a union of at most two intervals, and by our choice of \( \delta_n \), for each \( n > N \), the collection \( P_n \) is a \( \delta_n \)-fine subpartition of \( B_n \); in particular, \( B_n = C_n \cup \bigcup_{k \in K_n} A_k \).
As \( h = f \) on \( G \) and \( h = 0 \) on \( A - G \), we have

\[
|\sigma(f, P) - I| \leq |\sigma(g, P) - (L) \int_s f + \sum_{n=1}^N \int_{B_n} f| + |\sigma(h, P) - \sum_{n>N} \int_{B_n} f|
\]

\[
< \frac{\varepsilon}{3} + \left( \sum_{n \geq N} \sum_{k \in K_n} f(x_k) |A_k| - \sum_{n \geq N} \sum_{k \in K_n} \int_{A_k} f + \sum_{n \geq N} \int_{C_n} f \right)
\]

\[
\leq \frac{\varepsilon}{3} + \sum_{n \geq N} \sum_{k \in K_n} \left| f(x_k) |A_k| - \int_{A_k} f \right| + \sum_{n \geq N} \int_{C_n} f
\]

\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \sum_{n \geq N} 2^{-n} + 2 \sum_{n \geq N} W_n < \varepsilon,
\]

and the lemma is proved.

9. THEOREM. If \( A \) is an interval, then \( \mathcal{R}_*(A) = \mathcal{H}(A) \).

PROOF. In view of Theorem 2, it suffices to show that \( \mathcal{D}(A) \subset \mathcal{R}_*(A) \). However, by means of Proposition 4 and Lemmas 7 and 8, this follows readily from the constructive definition of the Denjoy-Perron integral (see [S, Chapter VIII, §5] or [N, Chapter XVI, §§6 and 7]).

10. REMARK. The proof of Theorem 9 does not generalize to higher dimensions (cf. Remark 6). Indeed, the proof is based on the possibility of obtaining the Henstock-Kurzweil integral by the Denjoy transfinite process, for which no satisfactory analogue in higher dimensions is known. Thus it is an open question whether Theorem 9 holds for the higher dimensional Henstock-Kurzweil integral (see [Mi]), or for its generalizations defined in [M, JKS, and P3].

REFERENCES


