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SINGULAR INTEGRALS IN PRODUCT DOMAINS AND THE METHOD OF ROTATIONS

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ABSTRACT. Singular integrals with kernels of the form $K(x,y)$ where K satisfies conditions to be a bounded singular integral operator in each of its variables have been much studied lately. In this paper we use the classical method of rotations to give a proof that kernels of the form $K(x,y) = \Omega(x,y)/|x|^n|y|^m$ correspond to bounded singular integral operators.

The purpose of this paper is to use the method of rotations to give a simple proof that Calderón-Zygmund type operators when generalized to product domains are bounded operators. In particular we consider kernels of the type

$$K(x,y) = \frac{\Omega(x,y)}{|x|^n|y|^m} \tag{1}$$

for $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$ and Ω satisfying certain conditions (which make it a C-Z kernel in each variable.) We are asking if $\|K * f\|_p \le C_p\|f\|_p$.

If $\Omega(x,y) = \Omega_1(x)\Omega_2(y)$ where Ω_1 and Ω_2 correspond to bounded operators on L^p then we can simply iterate one variable methods. In the case above this approach does not work. Kernels $K(x,y)$ not of the form of (1) but satisfying size and smoothness conditions like those of $1/xy$ have been much studied lately (see [**2**, **3**]). The kernels we will study in this paper are less general but can be handled entirely with single variable methods.

Before proceeding I want to thank Alberto Torchinsky for suggesting this approach.

We will proceed to use the method of rotations by studying even and odd kernels.

THEOREM 1. *Let $K(x,y) = \Omega(x,y)/|x|^n|y|^m$, Ω odd in both variables, homogeneous of degree zero and $\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}} |\Omega(x',y')|\,dx'\,dy' < \infty$. (Here Σ_{n-1} denotes the unit sphere in $\mathbf{R}^n$ and $x' = x/|x|$.)*

If $T_{\varepsilon,\eta}(f)(x,y) = \int_{|s|>\varepsilon}\int_{|t|>\eta} f(x-s,y-t)K(s,t)\,ds\,dt$ then

$$\left\|\sup_{\varepsilon,\eta}|T_{\varepsilon,\eta}(f)|\right\|_p \le C_p\|f\|_p, \qquad 1 < p < \infty.$$

PROOF. Using polar coordinates, let $s = r_1\ s'$, $t = r_2t'$, then

$$\begin{aligned} K_{\varepsilon,\eta} * f(x,y) &= \int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}} \Omega(s',t')\int_\varepsilon^\infty\int_\eta^\infty \frac{f(x-r_1s',y-r_2t')}{r_1r_2}\,dr_1\,dr_2\,ds'\,dt' \\ &= -\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}} \Omega(s',t')\int_\varepsilon^\infty\int_\eta^\infty \frac{f(x+r_1s',y-r_2t')}{r_1r_2}\,dr_1\,dr_2\,ds'\,dt', \end{aligned}$$

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since Ω is odd in the first variable. So the above expression equals

$$\frac{1}{2}\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}\Omega(s',t') \times \int_{\varepsilon}^{\infty}\int_{\eta}^{\infty}\frac{f(x-r_1s',y-r_2t')-f(x+r_1s',y-r_2t')}{r_1r_2}\,dr_1\,dr_2\,ds'\,dt'$$
$$=\frac{1}{2}\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}\Omega(s',t')\int_{|r_1|>\varepsilon}\int_{\eta}^{\infty}\frac{f(x-r_1s',y-r_2t')}{r_1r_2}\,dr_1\,dr_2\,ds'\,dt'.$$

Doing the same in the second variable we obtain

$$\frac{1}{4}\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}\Omega(s',t')\int_{|r_1|>\varepsilon}\int_{|r_2|>\eta}\frac{f(x-r_1s',y-r_2t')}{r_1r_2}\,dr_1\,dr_2\,ds'\,dt'.$$

Let S be the hyperplane perpendicular to s', and T to t'. Let $x = z+\lambda s'$ with $z\in S$, $y = w+\mu t'$ with $w\in T$. Then

$$=\frac{1}{4}\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}\Omega(s',t') \times \int_{|r_1|>\varepsilon}\frac{1}{r_1}\left(\int_{|r_2|>\eta}\frac{f(z+(\lambda-r)s',w+(\mu-r)t')}{r_2}\,dr_2\right)dr_1\,ds'\,dt'.$$

So

$$\left\|\sup_{\varepsilon,\eta}|T_{\varepsilon,\eta}(f)|\right\|_p \le \int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}|\Omega(s',t')| \times\left(\int_T\int_S\int_{\mathbf{R}}\int_{\mathbf{R}}\sup_{\varepsilon>0}\left|\int_{|r_1|>\varepsilon}\frac{1}{r_1}\right.\right.$$
$$\times\left(\sup_{\eta>0}\left|\int_{|r_2|>\eta}\frac{f(z+(\lambda-r_1)s',w+(\mu-r_2)t')}{r_2}\right)dr_2\right|dr_1\Big|^p\,d\lambda\,d\mu\,dw\,dz\Bigg)^{1/p}ds'\,dt'$$
$$\le\frac{C_p}{4}\int_{\Sigma_{n-1}}\int_{\Sigma_{m-1}}|\Omega(s',t')|\,ds'\,dt'\|f\|_p\le C_p\|f\|_p,$$

using the boundedness of the Maximal Hilbert transform twice. □

Exploiting this method we can in fact obtain

THEOREM 2. *Let $K(x,y)=\Omega(x',y')/|x|^n|y|^m$, where Ω is odd in the x'-variable. Let $K_x(y)=K(x,y)$ and $T_\varepsilon^x(f)(y)=\int_{|t|>\varepsilon}f(x,y-t)K_x(t)\,dt$. If*

$$(*)\qquad \|\sup|T_\varepsilon^x(f)|\|_p\le C_p\|f\|_p,$$

C_p independent of x (i.e., K is a bounded C-Z kernel in y independent of x), then

$$\left\|\sup_{\varepsilon,\eta}|T_{\varepsilon,\eta}(f)|\right\|_p\le C_p\|f\|_p.$$

PROOF. Proceeding as above, but in the x-variable only,

$$K_{\varepsilon,\eta}*f(x,y)=\frac{1}{2}\int_{\Sigma_{n-1}}\int_{|t|>\eta}\frac{\Omega(s',t)}{|t|^m}\int_{|r|>\varepsilon}\frac{f(x-rs',y-t)}{r}\,dr\,dt\,ds'.$$

Let $x = z + \lambda s'$, $x \in S$, where S is perpendicular to s',

$$= \frac{1}{2}\int_{\Sigma_{n-1}}\int_{|t|>\eta}\frac{\Omega(s',t)}{|t|^m}\int_{|r|>\varepsilon}\frac{f(z-(\lambda-r)s',y-t)}{r}\,dr\,dt\,ds'.$$

So

$$\left\|\sup_{\varepsilon,\eta}|T_{\varepsilon,\eta}|\right\|_p$$

$$\leq \frac{1}{2}\int_{\Sigma_{n-1}}\left(\int_S\int_{\mathbf{R}}\int_{\mathbf{R}^m}\sup_{\eta>0}\left|\int_{|t|>\eta}\frac{\Omega(s',t)}{t}\right.\right.$$

$$\left.\left.\times\sup_{\varepsilon>0}\left|\int_{|r|>\varepsilon}\frac{f(z-(\lambda-r)s',y-t)}{r}\,dr\right|\,dt\right|^p dy\,d\lambda\,dz\right)^{1/p}ds'.$$

Now using the assumption $(*)$, that our operator is bounded as an operator acting only in the second variable, we have

$$\leq C_p\int_{\Sigma_{n-1}}\left(\int_{\mathbf{R}^m}\int_S\int_{\mathbf{R}}\sup_{\varepsilon>0}\left|\int_{|r|>\varepsilon}\frac{f(z-(\lambda-r)s',y)}{r}\,dr\right|^p d\lambda\,dz\,dy\right)^{1/p}ds'$$

$$\leq C_p\int_{\Sigma_{n-1}}\left(\int_{\mathbf{R}^m}\int_S\int_{\mathbf{R}}|f(x,y)|^p\,dx\,dy\right)^{1/p}ds' = C_p\left(\int_{\Sigma_{n-1}}ds'\right)\|f\|_p. \quad \square$$

THEOREM 3. *Again let $K(x,y) = \Omega(x,y)/|x|^n|y|^m$, Ω homogeneous of degree zero in each variable, $\int_{\Sigma_{n-1}}\Omega(s,t)\,ds = 0$ a.e. in t, $\int_{\Sigma_{m-1}}\Omega(s,t)\,dt = 0$ a.e. in s, $(\int_{\Sigma_{n-1}}|\Omega(s,t)|^2\,ds)^{1/2} < C$ independent of t, and $(\int_{\Sigma_{m-1}}|\Omega(s,t)|^2\,dt)^{1/2} < C$ independent of s. Then $\|K * f\|_p \leq C_p\|f\|_p$.*

PROOF. We may assume Ω is even in both variables (since the hypotheses assure that both the previous theorems hold) and that $f = \rho$, a testing function. Let y be fixed, $K_y(x) = \Omega(x',y')/|x|^n$, $T_y(\rho) = K_y * \rho$ and R_i be the ith Riesz Transformation in x. Then [**4**, p. 225] shows that R_iT_y is essentially an odd C-Z operator. In fact, it is shown that $(R_iT_y)\hat{} = (J_i^y)\hat{}$ where $J_i^y(x) = \omega_{y'}(x')/|x|^n$, $\omega \in L^2(\Sigma_{n-1},dx)$ and $\omega_y(x)$ is odd in x.

If we set

$$J_i(x,y) = \frac{J_i^y(x)}{|y|^m} = \frac{\omega(x',y')}{|x|^n|y|^m}, \quad \text{where } \omega(x',y') = \omega_{y'}(x'),$$

then J_i is an odd C-Z integral operator in the x-variable.

In the y-variable we see that J_i is a C-Z operator (as expected) since

$$[(R_jT)\rho]\hat{} = -i\frac{x_j}{|x|}\Omega_0\left(\frac{x}{|x|},\frac{y}{|y|}\right)\hat{\rho}$$

where $\Omega_0 = \widehat{K}$ (see [**4**]). In the y-variable this acts exactly as did $K * \rho$ (up to a constant C_x, $|C_x| < 1$) and so is still a C-Z operator.

So by Theorem 2 above, $\|J_i * \rho\|_p \leq C\|\rho\|_p$.

It follows that

$$\|K * \rho\|_p = \|\Sigma R_i * J_i * \rho\|_p \leq \sum\|R_i * (J_i * \rho)\|_p \leq C\|\rho\|_p. \quad \square$$

There is a weighted version of this as well.

For one variable $x \in \mathbf{R}^n$, a function $w(x) > 0$ is called an A_p-weight, $1 < p < \infty$, if it satisfies

$$\frac{1}{|Q|}\int_Q w(x)\,dx \left(\frac{1}{|Q|}\int_Q [w(x)]^{-1/(p-1)}\,dx\right)^{p-1} \leq B,$$

for some $B < \infty$ and all cubes Q in $\mathbf{R}^n$.

The following theorem is well known.

THEOREM. *If w is an A_p-weight and K is a standard C-Z kernel then*

$$\int_{\mathbf{R}^n} |f * K(y)|^p w(y)\,dy \leq C(p,B)\int_{\mathbf{R}^n} |f(y)|^p w(y)\,dy.$$

See [**1**] *for results on weights.*

In the work above we used only that $T = K * f$ was bounded in L^p-norm in each variable separately. Thus if $w(x,y)$ is an A_p-weight in each variable, i.e., if $w(x,y) > 0$, $x \to w(x,y)$ is an A_p-weight with B independent of y and similarly for $y \to w(x,y)$ then following the method of proof above we have the following.

THEOREM 4. *If w is as above and K satisfies the conditions of Theorem* 3 *then*

$$\left(\int_{\mathbf{R}^n\times\mathbf{R}^m} |K * f(y)|^p w(y)\,dy\right)^{1/p} \leq C(p,A)\left(\int_{\mathbf{R}^n\times\mathbf{R}^m} |f(y)|^p w(y)\,dy\right)^{1/p}.$$

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