SINGULAR INTEGRALS IN PRODUCT DOMAINS
AND THE METHOD OF ROTATIONS
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ABSTRACT. Singular integrals with kernels of the form $K(x, y)$ where $K$ satisfies conditions to be a bounded singular integral operator in each of its variables have been much studied lately. In this paper we use the classical method of rotations to give a proof that kernels of the form $K(x, y) = \frac{\Omega(x, y)}{|x|^n|y|^m}$ correspond to bounded singular integral operators.

The purpose of this paper is to use the method of rotations to give a simple proof that Calderón-Zygmund type operators when generalized to product domains are bounded operators. In particular we consider kernels of the type

$$K(x, y) = \frac{\Omega(x, y)}{|x|^n|y|^m}$$

for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $\Omega$ satisfying certain conditions (which make it a C-Z kernel in each variable.) We are asking if $\|K \ast f\|_p \leq C_p \|f\|_p$.

If $\Omega(x, y) = \Omega_1(x)\Omega_2(y)$ where $\Omega_1$ and $\Omega_2$ correspond to bounded operators on $L^p$ then we can simply iterate one variable methods. In the case above this approach does not work. Kernels $K(x, y)$ not of the form of (1) but satisfying size and smoothness conditions like those of $1/xy$ have been much studied lately (see [2, 3]). The kernels we will study in this paper are less general but can be handled entirely with single variable methods.

Before proceeding I want to thank Alberto Torchinsky for suggesting this approach.

We will proceed to use the method of rotations by studying even and odd kernels.

**THEOREM 1.** Let $K(x, y) = \frac{\Omega(x, y)}{|x|^n|y|^m}$, $\Omega$ odd in both variables, homogeneous of degree zero and $\int_{\Sigma_{n-1}} \int_{\Sigma_{m-1}} |\Omega(x', y')| \, dx' \, dy' < \infty$. (Here $\Sigma_{n-1}$ denotes the unit sphere in $\mathbb{R}^n$ and $x' = x/|x|$.)

If $T_{e, n}(f)(x, y) = \int_{|s| > \varepsilon} \int_{|t| > \eta} f(x - s, y - t)K(s, t) \, ds \, dt$ then

$$\left\| \sup_{e, \eta} |T_{e, n}(f)| \right\|_p \leq C_p \|f\|_p, \quad 1 < p < \infty.$$  

**PROOF.** Using polar coordinates, let $s = r_1 s', t = r_2 t'$, then

$$K_{e, n} \ast f(x, y) = \int_{\Sigma_{n-1}} \int_{\Sigma_{m-1}} \Omega(s', t') \int_{\varepsilon}^{\infty} \int_{\eta}^{\infty} \frac{f(x - r_1 s', y - r_2 t')}{{r_1 r_2}} \, dr_1 \, dr_2 \, ds' \, dt'$$

$$= - \int_{\Sigma_{n-1}} \int_{\Sigma_{m-1}} \Omega(s', t') \int_{\varepsilon}^{\infty} \int_{\eta}^{\infty} \frac{f(x + r_1 s', y - r_2 t')}{{r_1 r_2}} \, dr_1 \, dr_2 \, ds' \, dt',$$

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since $\Omega$ is odd in the first variable. So the above expression equals
\[
\frac{1}{2} \int_{\Sigma_{n-1}} \int_{\Sigma_{m-1}} \Omega(s', t') \\
\times \int_{\eta}^{\infty} \int_{\eta}^{\infty} \frac{f(x - r_1 s', y - r_2 t') - f(x + r_1 s', y - r_2 t')}{r_1 r_2} \, dr_1 \, dr_2 \, ds' \, dt'.
\]
Doing the same in the second variable we obtain
\[
\frac{1}{4} \int_{\Sigma_{n-1}} \int_{\Sigma_{m-1}} \Omega(s', t') \int_{|r_1| > \eta} \int_{|r_2| > \eta} \frac{f(x - r_1 s', y - r_2 t')}{r_1 r_2} \, dr_1 \, dr_2 \, ds' \, dt'.
\]
Let $S$ be the hyperplane perpendicular to $s'$, and $T$ to $t'$. Let $x = z + \lambda s'$ with $z \in S$, $y = w + \mu t'$ with $w \in T$. Then
\[
\frac{1}{4} \int_{\Sigma_{n-1}} \int_{\Sigma_{m-1}} \Omega(s', t') \\
\times \int_{|r_1| > \eta} \int_{|r_2| > \eta} \frac{f(z + (\lambda - \mu) s', w + (\mu - \eta) t')}{r_2} \, dr_1 \, ds' \, dt'.
\]
So
\[
\left\| \sup_{\epsilon, \eta} |T_{\epsilon, \eta}(f)| \right\|_p \leq \int_{\Sigma_{n-1}} \int_{\Sigma_{m-1}} |\Omega(s', t')| \\
\times \left( \int_{T} \int_{S} \int_{R} \sup_{\epsilon > 0} \int_{|r_1| > \eta} \frac{1}{r_1} \right) \\
\times \left( \sup_{\eta > 0} \int_{|r_2| > \eta} \frac{f(z + (\lambda - \mu) s', w + (\mu - \eta) t')}{r_2} \, dr_2 \bigg| \, dr_1 \bigg| d\lambda \, d\mu \, dw \, dz \right)^{1/p} \, ds' \, dt' \\
\leq \frac{C_p}{4} \int_{\Sigma_{n-1}} \int_{\Sigma_{m-1}} |\Omega(s', t')| \, ds' \, dt' \|f\|_p \leq C_p \|f\|_p,
\]
using the boundedness of the Maximal Hilbert transform twice. □

Exploiting this method we can in fact obtain

**Theorem 2.** Let $K(x, y) = \Omega(x', y')/|x|^m|y|^m$, where $\Omega$ is odd in the $x'$-variable. Let $K_x(y) = K(x, y)$ and $T_{x, y}^e(f)(y) = \int_{|t| > \eta} f(x, y - t)K_x(t) \, dt$. If
\[
(*) \quad \| \sup_{\epsilon, \eta} |T_{x, y}^e(f)| \|_p \leq C_p \|f\|_p,
\]
$C_p$ independent of $x$ (i.e., $K$ is a bounded C-Z kernel in $y$ independent of $x$), then
\[
\left\| \sup_{\epsilon, \eta} |T_{x, y}^e(f)| \right\|_p \leq C_p \|f\|_p.
\]

**Proof.** Proceeding as above, but in the $x$-variable only,
\[
K_{x, y} \ast f(x, y) = \frac{1}{2} \int_{\Sigma_{n-1}} \int_{|t| > \eta} \frac{\Omega(s', t)}{|t|^m} \int_{|r| > \epsilon} \frac{f(x - rs', y - t)}{r} \, dr \, dt \, ds'.
\]
Let \( x = z + \lambda s' \), \( x \in S \), where \( S \) is perpendicular to \( s' \),
\[
\frac{1}{2} \int_{\Sigma_{n-1}} \int_{|t|>\eta} \frac{\Omega(s', t)}{|t|^m} \int_{|r|>\epsilon} f(z - (\lambda - r)s', y - t) \frac{dr}{r} dt ds'.
\]
So
\[
\left\| \sup_{\epsilon, \eta} |T_{\epsilon, \eta}| \right\|_p
\leq \frac{1}{2} \int_{\Sigma_{n-1}} \left( \int_S \int_{R^m} \sup_{\eta>0} \int_{|t|>\eta} \frac{\Omega(s', t)}{t} \right) \left( \int_{|r|>\epsilon} f(z - (\lambda - r)s', y - t) \frac{dr}{r} dt \right)^p dy d\lambda dz ds'.
\]
Now using the assumption (*), that our operator is bounded as an operator acting only in the second variable, we have
\[
\leq C_p \int_{\Sigma_{n-1}} \left( \int_S \int_{R^m} \sup_{\epsilon>0} \left( \int_{|r|>\epsilon} f(z, y) \frac{dr}{r} \right)^p dy d\lambda dz \right)^{1/p} ds'.
\]

**THEOREM 3.** Again let \( K(x, y) = \Omega(x, y)/|x|^n|y|^m \), \( \Omega \) homogeneous of degree zero in each variable, \( \int_{\Sigma_{n-1}} \Omega(s, t) ds = 0 \) a.e. in \( t \), \( \int_{\Sigma_{n-1}} \Omega(s, t) dt = 0 \) a.e. in \( s \), \( (\int_{\Sigma_{n-1}} |\Omega(s, t)|^2 ds)^{1/2} < C \) independent of \( t \), and \( (\int_{\Sigma_{n-1}} |\Omega(s, t)|^2 dt)^{1/2} < C \) independent of \( s \). Then \( \| K * f \|_p \leq C_p \| f \|_p \).

**PROOF.** We may assume \( \Omega \) is even in both variables (since the hypotheses assure that both the previous theorems hold) and that \( f = \rho \), a testing function. Let \( y \) be fixed, \( K_y(x) = \Omega(x', y')/|x|^n \), \( T_y(\rho) = K_y * \rho \) and \( R_i \) be the \( i \)th Riesz Transformation in \( y \). Then [4, p. 225] shows that \( R_i T_y \) is essentially an odd C-Z operator. In fact, it is shown that \( (R_i T_y)^\sim = (J_i^y)^\sim \) where \( J_i^y(x) = \omega_i(x')/|x|^n \), \( \omega \in L^2(\Sigma_{n-1}, dx) \) and \( \omega_y(x) \) is odd in \( x \).

If we set
\[
J_i(x, y) = \frac{J_i^y(x)}{|y|^m} = \frac{\omega(x', y')}{|x|^n|y|^m}, \quad \text{where } \omega(x', y') = \omega_y(x'),
\]
then \( J_i \) is an odd C-Z integral operator in the \( x \)-variable.

In the \( y \)-variable we see that \( J_i \) is a C-Z operator (as expected) since
\[
[(R_j T)\rho]^\sim = -i \frac{x_j}{|x|} \Omega_0 \left( \frac{x}{|x|}, \frac{y}{|y|} \right) \hat{\rho}
\]
where \( \Omega_0 = \hat{K} \) (see [4]). In the \( y \)-variable this acts exactly as did \( K * \rho \) (up to a constant \( C_z \), \( |C_z| < 1 \)) and so is still a C-Z operator.

So by Theorem 2 above, \( \| J_i * \rho \|_p \leq C \| \rho \|_p \).

It follows that
\[
\| K * \rho \|_p = \| R_i * J_i * \rho \|_p \leq \sum \| R_i * (J_i * \rho) \|_p \leq C \| \rho \|_p.
\]
There is a weighted version of this as well.

For one variable $x \in \mathbb{R}^n$, a function $w(x) > 0$ is called an $A_p$-weight, $1 < p < \infty$, if it satisfies

$$
\frac{1}{|Q|} \int_Q w(x) \, dx \left( \frac{1}{|Q|} \int_Q [w(x)]^{-(p-1)/p} \, dx \right)^{p-1} \leq B,
$$

for some $B < \infty$ and all cubes $Q$ in $\mathbb{R}^n$.

The following theorem is well known.

**THEOREM.** If $w$ is an $A_p$-weight and $K$ is a standard $C$-$Z$ kernel then

$$
\int_{\mathbb{R}^n} |f * K(y)|^p w(y) \, dy \leq C(p, B) \int_{\mathbb{R}^n} |f(y)|^p w(y) \, dy.
$$

See [1] for results on weights.

In the work above we used only that $T = K * f$ was bounded in $L^p$-norm in each variable separately. Thus if $w(x, y)$ is an $A_p$-weight in each variable, i.e., if $w(x, y) > 0$, $x \rightarrow w(x, y)$ is an $A_p$-weight with $B$ independent of $y$ and similarly for $y \rightarrow w(x, y)$ then following the method of proof above we have the following.

**THEOREM 4.** If $w$ is as above and $K$ satisfies the conditions of Theorem 3 then

$$
\left( \int_{\mathbb{R}^n \times \mathbb{R}^m} |K * f(y)|^p w(y) \, dy \right)^{1/p} \leq C(p, A) \left( \int_{\mathbb{R}^n \times \mathbb{R}^m} |f(y)|^p w(y) \, dy \right)^{1/p}.
$$

**REFERENCES**


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