AN EXTENSION OF THE CLOSED UNBOUNDED FILTER

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ABSTRACT. A natural extension of the closed unbounded filter is introduced. This extension coincides with the closed unbounded filter on uncountable, regular cardinals $\kappa$, but in general does not for $P_\kappa \lambda$ and $[\lambda]^{\kappa^+}$.

Henceforth, $\kappa$ will be a regular, uncountable cardinal, unless specified otherwise. A closed unbounded subset of $\kappa$ is a cofinal subset of $\kappa$ which contains the supremum of all increasing sequences from the subset of length less than $\kappa$. The collection of all closed unbounded subsets of $\kappa$ generates a $\kappa$-complete, normal filter over $\kappa$ called the club filter. The notion of a closed unbounded subset of a cardinal $\kappa$ has been generalized to the set of all subsets of $\lambda$ of cardinality less than $\kappa$, $P_\kappa \lambda$ (see [3]), and the set of all subsets of $\lambda$ of cardinality $\kappa$, $[\lambda]^{\kappa^+}$ (see [2]). In both instances the collection of closed unbounded sets generate $\kappa$-complete, fine, normal filters, the club filters. This paper introduces a natural extension of the club filter. This extension coincides with the club filter on $\kappa$, but in general does not for $P_\kappa \lambda$ and $[\lambda]^{\kappa^+}$.

The motivation for the filter arose from the desirable property of certain sequences (or proper chains in the case of $P_\kappa \lambda$) $\{p_\alpha : \alpha < \delta\}$ which satisfy

$$\left| \bigcup_{\alpha < \delta} p_\alpha \right| = \bigcup_{\alpha < \delta} |p_\alpha|,$$

where $|A|$ denotes the cardinality of $A$. Since

$$\left| \bigcup_{\alpha < \delta} p_\alpha \right| = |\delta| \bigcup_{\alpha < \delta} |p_\alpha|$$

this property is satisfied by sequences (or chains) $\{p_\alpha : \alpha < \delta\}$ which satisfy

$$|\delta| \leq \bigcup_{\alpha < \delta} |p_\alpha|.$$

A canonical example of a sequence which fails to satisfy this is $\{p_\alpha : \alpha < \delta\}$ where $p_\alpha = \alpha$ and $\delta = \omega_1$.

DEFINITION. A subset $b$ of $\kappa$ is said to be $L$-closed if for any sequence $\{p_\alpha : \alpha < \delta\} \subseteq b$ with $\delta < \kappa$ and $|\delta| \leq \bigcup_{\alpha < \delta} |p_\alpha|$, then $\bigcup_{\alpha < \delta} p_\alpha \in b$.

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An $L$-closed, unbounded subset of $\kappa$ is a cofinal subset of $\kappa$ which is $L$-closed. The verification that the $L$-closed, unbounded subsets of $\kappa$ generate a $\kappa$-complete, normal filter over $\kappa$, called the $L$-club filter, is a routine exercise once familiar with the details of generating the club filter over $\kappa$ (see [4]). The $L$-club filter and the club filter coincide on $\kappa$: Let $b \subset \kappa$ be closed unbounded. Then $b$ is $L$-closed unbounded. And if $b \subset \kappa$ is $L$-closed unbounded, consider

(i) if $\kappa$ is a limit cardinal

$$a = b \cap \{\alpha < \kappa: \alpha \text{ is a cardinal}\}.$$  

(ii) if $\kappa$ is a successor cardinal and $\kappa = \gamma^+$

$$a = (b \cap \kappa - \gamma).$$  

In both cases $a \subset b$ and $a$ is closed unbounded.

For $X \subset P_\kappa\lambda$, $X$ is unbounded if for any $x \in P_\kappa\lambda$, there exists a $y \in X$ such that $x \subset y$. And $X$ is closed if for any collection $\{p_\alpha: \alpha < \delta\} \subset X$ with $p_\alpha \subset p_{\alpha+1}$ for $\alpha < \delta$ (called a chain of subsets from $P_\kappa\lambda$) with $\delta < \kappa$, then $\bigcup_{\alpha < \delta} p_\alpha \in X$.

**DEFINITION.** A set $X \subset P_\kappa\lambda$ is said to be $L$-closed if whenever $\{p_\alpha: \alpha < \delta\} \subset X$ is a proper chain such that $|\delta| \leq \bigcup_{\alpha < \delta} |p_\alpha| < \kappa$, then $\bigcup_{\alpha < \delta} p_\alpha \in X$.

Let

$$L_\kappa\lambda = \{A \subset P_\kappa\lambda: \text{ there exists an } X \subset A \text{ which is } L\text{-closed unbounded}\}.$$  

**PROPOSITION 1.** $L_\kappa\lambda$ is a $\kappa$-complete, fine, normal filter over $P_\kappa\lambda$.

**PROOF.** The proof follows closely the proof that the closed unbounded sets on $P_\kappa\lambda$ generate the club filter (see [3]). The only modification for this proof requires that wherever a chain $\{p_\alpha: \alpha < \delta\}$ is constructed in the club filter proof, the chain must be made to satisfy $|\delta| \leq \bigcup_{\alpha < \delta} |p_\alpha|$. This presents little difficulty in any of the situations where chains are needed. The proof of the next proposition will show the ease at which a chain, in most of the circumstances required here, can be made to satisfy this condition.

A subset $D \subset P_\kappa\lambda$ is said to be directed if given $x, y \in D$, then there exists $z \in D$ such that $x \cup y \subset z$. A subset $A \subset P_\kappa\lambda$ is said to be closed under directed sets if given $D \subset X$ such that $|D| < \kappa$ and $D$ is directed, then $\bigcup D \in X$. It is a result of Solovay that $X$ is closed unbounded if and only if $X$ is closed under directed sets (see [5]). The analog here is the following:

**PROPOSITION 2.** An $L$-closed unbounded subset of $P_\kappa\lambda$ is closed under unions of directed sets $D$ where $|D| \leq \bigcup\{|p|: p \in D\}$.

**PROOF.** (This is basically Solovay’s proof with the required modifications.) Let $B$ be an $L$-closed unbounded subset of $P_\kappa\lambda$ and $D \subset B$ a directed set such that $|D| \leq \bigcup\{|p|: p \in D\}$. Only the case where $|D| > \aleph_0$ requires an adjustment. Assume if $D'$ is any directed subset of $B$, $|D'| < |D|$ and $|D'| \leq \bigcup\{|p|: p \in D\}$, then $\bigcup D' \in B$.

**CLAIM.** Given any set $X \subset D$, there exists a set $X^+$ such that

1. $X \subset X^+ \subset D$,
2. $|X^+| \leq |X| + \aleph_0 \leq \bigcup\{|p|: p \in X^+\}$ and
3. $X^+$ is directed.

**PROOF OF CLAIM.** See [5].
Well order $D$ by $p_0, p_1, \ldots, p_\alpha, \ldots$ where $\alpha < |D|$. Let

\begin{align*}
D_0 &= \{p_0\}, \\
D_1 &= \{D_0, p_1\}^+, \\
D_2 &= \{D_0, D_1, p_2, q_2\}^+ \text{ where } q_2 \in D \text{ such that } |q_2| \geq 2, \\
&\vdots \\
D_\alpha &= \{\{D_\beta\}_{\beta < \alpha}, p_\alpha, q_\alpha\}^+,
\end{align*}

where $q_\alpha \in D$ such that $|q_\alpha| \geq \alpha + \aleph_0$. (Note: such a $q_\alpha$ exists since $|D| \leq \bigcup \{|p| : p \in D\}$.) Now,

$$|D_\alpha| = \left|\{\{D_\beta\}_{\beta < \alpha}, p_\alpha, q_\alpha\}^+\right| \leq \alpha + \aleph_0 < |D|.$$

Hence,

$$|D_\alpha| \leq \bigcup \{|p| : p \in D_\alpha\}.$$

By our assumption, $\bigcup D_\alpha \in B$. So let

$$Q_\alpha = \bigcup D_\alpha \in B \quad \text{for all } \alpha < |D|.$$

Then $\{Q_\alpha : \alpha < |D|\} \subseteq B$ is a chain and

$$|D| \leq \bigcup \{\alpha^+: \alpha < |D|\} \leq \bigcup \{|q_\alpha| : \alpha < |D|\} \leq \bigcup \{|U D_\alpha| : \alpha < |D|\} = \bigcup \{|Q_\alpha| : \alpha < |D|\}.$$

Since $B$ is $L$-closed, $\bigcup D = \bigcup_{\alpha < |D|} Q_\alpha \in B$.

In contrast to the situation on cardinals where the $L$-club and club filters coincide, the next proposition shows that in general this not the case for $P_\kappa \lambda$ when a large cardinal assumption is made on $\kappa$.

**Proposition 3.** Assume $\kappa$ is a huge cardinal and $\lambda$ is any cardinal greater than $\kappa$ such that there exists a $\kappa$-complete, fine, normal ultrafilter over $[\lambda]^\kappa$, then there exists an $L$-closed unbounded subset of $P_\kappa \lambda$ which is not in the club filter.

**Proof.** Since $\lambda$ must be measurable (see [1 or 6]), a regular cardinal $\gamma$ can be chosen such that $\kappa < \gamma < \lambda$. Now $\lambda$ must be a regular cardinal so $\lambda$ can be partitioned into $\lambda$-many disjoint intervals of length $\gamma$. For $\alpha < \lambda$ denote the $\alpha$th such interval as $\gamma_\alpha$. Given any $\beta < \lambda$ let the $\gamma$th index of $\beta$, denoted $\gamma(\beta)$, be

$$\Gamma(\beta) = \text{order type of } \gamma_\alpha \cap \beta, \quad \text{where } \beta \in \gamma_\alpha.$$

For $x \in P_\kappa \lambda$ let

$$\Gamma''x = \{\Gamma(\beta) : \beta \in x\}.$$

Set

$$B = \{x \in P_\kappa \lambda : |\Gamma''x| = |x|\}.$$

**Claim 1.** $B$ is $L$-closed unbounded.

**Proof of Claim.** $B$ is readily seen to be unbounded, since for any $x \in P_\kappa \lambda$, $x \cup |x| \cup \omega \in B$.

Next, let $\{p_\alpha : \alpha < \delta\} \subseteq B$ be a chain of size $\delta < \kappa$, where $|\delta| \leq \bigcup_{\alpha < \delta} |p_\alpha|$. Suppose $|\bigcup_{\alpha < \delta} p_\alpha| > |\Gamma'' \bigcup_{\alpha < \delta} p_\alpha|$. Let $\beta = |\Gamma'' \bigcup_{\alpha < \delta} p_\alpha|$. So $|\bigcup_{\alpha < \delta} p_\alpha| > \beta$. 

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Since $\bigcup_{\alpha<\delta} |p_\alpha| = |\bigcup_{\alpha<\delta} p_\alpha| > \beta$ there exists $\alpha < \delta$ such that $|p_\alpha| > \beta$. But $|\Gamma''p_\alpha| = |p_\alpha|$, which is clearly false. Therefore, $\bigcup_{\alpha<\gamma} p_\alpha \in B$, and the claim is proved.

DiPrisco and Marek used the following type of construction to define their notion of closed unbounded sets on $[\lambda]^\kappa$ (see [2]):

For $B \subseteq P_\kappa \lambda$ from above, set

$$A_B = \left\{ p \in [\lambda]^\kappa : \text{there exists a chain } \{p_\alpha : \alpha < \kappa\} \subseteq B \text{ and } p = \bigcup_{\alpha<\kappa} p_\alpha \right\}.$$  

**CLAIM 2.** If $p \in A_B$, then $|\Gamma''p| = \kappa$.

**PROOF OF CLAIM.** Assume $|\Gamma''p| < \kappa$. Since $\kappa = |p| = |\bigcup_{\alpha<\kappa} p_\alpha| = \bigcup_{\alpha<\kappa} |p_\alpha|$ there exists $\alpha < \kappa$ such that $|p_\alpha| > |\Gamma''p|$. But this gives

$$|\Gamma''p_\alpha| = |p_\alpha| > |\Gamma''p| = \left| \Gamma'' \bigcup_{\alpha<\kappa} p_\alpha \right|,$$

which cannot be true.

Finally, assume there exists a closed unbounded set $C$ from $P_\kappa \lambda$ which is a subset of $B$. By a result in [2] if $U$ is any $\kappa$-complete, fine, normal ultrafilter over $[\lambda]^\kappa$ and $C$ is any closed unbounded subset of $P_\kappa \lambda$, then $A_C \in U$. So by our assumption, $A_B \in U$. That is, $\{p \in [\lambda]^\kappa : |\Gamma''p| = \kappa\} \in U$.

Let $j : V \to M$ be the canonical elementary embedding produced by the ultrafilter $U$ on $[\lambda]^\kappa$. Then, $A \in U$ iff $j''\lambda \in j(A)$. Hence,

$$M \models (j\Gamma)'''j''\lambda = j(\kappa)$$

by the above. However, the definition of $\Gamma$ and the elementarity of $j$ yield

$$M \models (j\Gamma)'''j''\lambda = j''\gamma.$$

But, this is a contradiction, since $\gamma < \lambda$.

This demonstrates that there are subsets of $P_\kappa \lambda$ which are $L$-closed unbounded but not closed. However, such subsets of $P_\kappa \kappa^+$ do not exist.

Whether or not an $L$-closed unbounded subset of $P_\kappa \lambda$ can be constructed which does not contain a closed unbounded subset, without first assuming the existence of a huge cardinal, is open. However, there exist examples of $L$-closed unbounded subsets of $P_\kappa \lambda$ which are not closed unbounded. The following was provided by C. A. DiPrisco:

$$E = \{p \in P_\kappa \lambda : |p \cap \kappa| = |p - \kappa| \}.$$  

The problem remains to determine whether or not such sets are in the club filter on $P_\kappa \lambda$.

**REFERENCES**


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