CONGRUENCE AND ONE-DIMENSIONALITY OF METRIC SPACES

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ABSTRACT. Two subsets $A$ and $B$ of a metric space $(X, d)$ are said to be congruent if there is a bijection between them which preserves the distance $d$. We show that if a separable locally compact metric space is such that no distinct subsets of cardinality 3 are congruent then its dimension is $\leq 1$. We also show that the real line $\mathbb{R}$ can be given a compatible metric with this property.

1. Introduction. We say that two subsets $A, B$ of a metric space $(X, d)$ are congruent if there exists a bijection between them which preserves the distance $d$. Using the notion of congruence our previous result [2] says that a nonempty separable metrizable space is zero-dimensional if and only if it admits a metric relative to which no two distinct sets of cardinality 2 are congruent. Our objective in this note is to explore the case where the cardinality of the sets mentioned above is 3. We prove

THEOREM 1. If $X$ is a locally compact separable metric space having a metric $d$ such that no two distinct subsets of $X$ of cardinality 3 are congruent relative to $d$, then $\dim(A) < 1$.

THEOREM 2. The real line $\mathbb{R}$ can be given a compatible metric $d$ so that $(\mathbb{R}, d)$ contains no two distinct congruent subsets of cardinality 3.

2. Proof of the theorems. If $x_1, x_2 \in X$ are two distinct points of a metric space $(X, d)$ we denote by $B(x_1, x_2)$ the bisector set defined as $\{y : y \in X$ and $d(x_1, y) = d(x_2, y)\}$ (cf. [3, 4, or 5]).

LEMMA 2.1. Let $X$ be a separable metrizable space. Then, $\dim(X) < n$ if and only if $X$ has an admissible totally bounded metric $d$ such that if $B$ is a $d$-bisector set in $X$ then $\dim(B) < n - 1$.

PROOF. This is Theorem 1 of [4].

As a corollary to this lemma we prove

LEMMA 2.2. Let $(X, d)$ be a totally bounded metric space containing no two distinct congruent sets of cardinality 3. Then $\dim(X) \leq 1$.

PROOF. If $X$ is empty or if it contains only one point then $\dim(X) \leq 0$. So assume $X$ contains at least two points and let $B = B(x_1, x_2)$ be a bisector set of...
We observe that the set $B$ cannot contain more than one point since if $y_1, y_2 \in B$ were two distinct points then \{$(x_1, y_1, y_2)$ and $(x_2, y_1, y_2)$\} would be two distinct congruent sets of cardinality 3 which is contrary to our hypothesis. Thus we have that $\dim(B) \leq 0$ for every bisector set of $X$ and Lemma 2.1 implies that $\dim(X) \leq 1$ which was to be proved.

The proof of our Theorem 1 now follows as a corollary to Lemma 2.2. Since $(X, d)$ is a locally compact separable metric space each point $x \in X$ is contained in a precompact open subset $U \subseteq X$ so that the restriction of $d$ to $U$ is totally bounded. Since the dimension is a local property we conclude that $\dim(X) \leq 1$.

To prove our Theorem 2 we need some notation. We denote by $\mathbb{R}^2$ the Euclidean plane with the usual Euclidean metric on it. Let $I$ denote the group of all orientation preserving isometries of $\mathbb{R}^2$, i.e., the proper motions and let $I^*$ be the group of all isometries of $\mathbb{R}^2$ including reflections. Let $P$ denote the graph $\{(x, x^2) : x \in \mathbb{R}\}$ of the parabola $y = x^2$ in $\mathbb{R}^2$ and let $P^+ \subseteq \mathbb{R}^2$ denote the following subset of it: \{(x, x^2) : x > 0\}. We observe that $P^+$ is homeomorphic to $\mathbb{R}$ and we shall prove that it has the desired property relative to the Euclidean metric of $\mathbb{R}^2$.

**Lemma 2.3.** Let $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ be two distinct subsets of $P^+$ of cardinality 3. Then these sets cannot be congruent.

**Proof.** We give an indirect proof, assuming they are congruent. Then, as we know (cf. [1]) the isometry between $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ extends uniquely to an element $T \in I^*$. Now we distinguish two cases: Case (1) where $T$ is a proper motion. Let the points $A_i = (x_i, x_i^2)$ and $B_j(y_j, y_j^2)$ be indexed so that for their first coordinates $x_i$ and $y_j$ we have $0 < x_1 < x_2 < x_3$ and $0 < y_1 < y_2 < y_3$ respectively. Then, since $T$ is order-preserving we have: $B_i = TA_i$ for $i = 1, 2, 3$. Without loss of generality we may assume $x_1 \leq y_1$ from which one easily deduces that $x_2 \leq y_2$ and $x_3 \leq y_3$. An elementary computation shows that the negatively taken tangent of the angle at the vertex $A_2$ of the triangle $\{A_1, A_2, A_3\}$ equals

$$a_{13}[1 + (x_1 + x_2)(x_2 + x_3)(1 + x_1^2 + 2x_1x_3 + x_3^2)]^{-1}$$

where $a_{13} = [(x_3 - x_1)^2 + (x_3^2 - x_1^2)^2]^{1/2}$ is the distance between $A_1$ and $A_3$. Since both of these quantities are preserved under $T$ we obtain the equality

$$1 + (x_1 + x_2)(x_2 + x_3)(1 + x_1^2 + 2x_1x_3 + x_3^2) = 1 + (y_1 + y_2)(y_2 + y_3)(1 + y_1^2 + 2y_1y_3 + y_3^2).$$

But since the expression on the left is an increasing function of $x_1, x_2$ and $x_3$ and since $y_1 \geq x_1, y_2 \geq x_2, y_3 \geq x_3$ the equality would imply $x_i = y_i$ for $i = 1, 2, 3$, which is impossible since the sets $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ are distinct. So it remains to investigate Case (2), where $T$ is a reflection. Let $L$ be the line in $\mathbb{R}^2$ which is pointwise invariant under $T$. We observe that the set $\{A_1, A_2, A_3, B_1, B_2, B_3\}$ has cardinality at least 4 and that it is contained in the intersection $P^+ \cap TP^+$. We also observe that the line $L$ would intersect the parabola $P$ in two points $C_1, C_2$ where $C_1 \in P^+$ and $C_2 \notin P^+$. Thus, the set $\{A_1, A_2, A_3, B_1, B_2, B_3, C_2\}$ with cardinality at least 5 would be in the intersection $P \cap TP$ which contradicts the fact that two quadratic curves have at most 4 common points. So we arrived at the desired contradiction in both cases which concludes the proof of our lemma.

Identifying the line $\mathbb{R}$ with the set $P^+$ and taking for $d$ the restriction of the Euclidean metric of $\mathbb{R}^2$ to $P^+$ we obtain the proof of our Theorem 2.
REMARK. The fact that \( \mathbb{R} \) can be remetrized in such a manner that no two distinct sets of cardinality 3 are congruent supports our belief that also the converse of Theorem 1 is true, namely that if a separable metrizable space has dimension \( \leq 1 \) then it has a metric with this property. One way to show it would be to prove that the universal 1-dimensional subset of \( \mathbb{R}^3 \), the Sierpiński cube, has such a metric. In [3] is shown that there exists a subset of \( \mathbb{R} \) containing no two distinct congruent subsets of cardinality 2 which is homeomorphic to the Cantor set \( C \). We conjecture that there exists a set \( S^* \subseteq \mathbb{R}^3 \) homeomorphic to the Sierpiński set \( S \) such that for every nonidentical isometry \( T \) of \( \mathbb{R}^3 \) the cardinality of the intersection \( S^* \cap TS^* \) is at most 2 if \( T \) is a proper motion and at most 3 if it is a reflexion. This would imply that the set \( S^* \) contains no two distinct congruent subsets of cardinality 3.

REFERENCES

1. L. M. Blumenthal, Theory and applications of distance geometry, Oxford Univ. Press, 1953.

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